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$F(s) = L\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\,dt$

Eulers Formula:

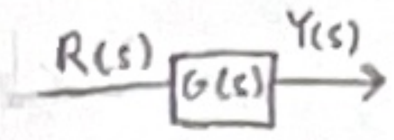
$e^{j\theta} = \cos\theta - j\sin\theta$

$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$

$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

$\mathrm{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$

Transfer Function:



$Y(s) = G(s) \cdot R(s)$

Laplace Transform:

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$r(t)$	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$e^{-at}\sin(bt)$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos(bt)$	$\frac{s+a}{(s+a)^2+b^2}$
$t^n f(t)$	$(-1)^n \frac{d}{ds^n}(F(s))$

Properties:

① Time Delay, $f(t-T)u(t-T) \longleftrightarrow e^{-Ts}F(s)$

② Differentiation,

$f''(t) + f'(t) + f(t)$

$\updownarrow$

$s^2F(s) - sf(0) - f'(0) + sF(s) - f(0) + F(s)$

③ Final Value Thrm, "if all the poles of sF(s) are in the LHP with possibly one simple pole at the origin"

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

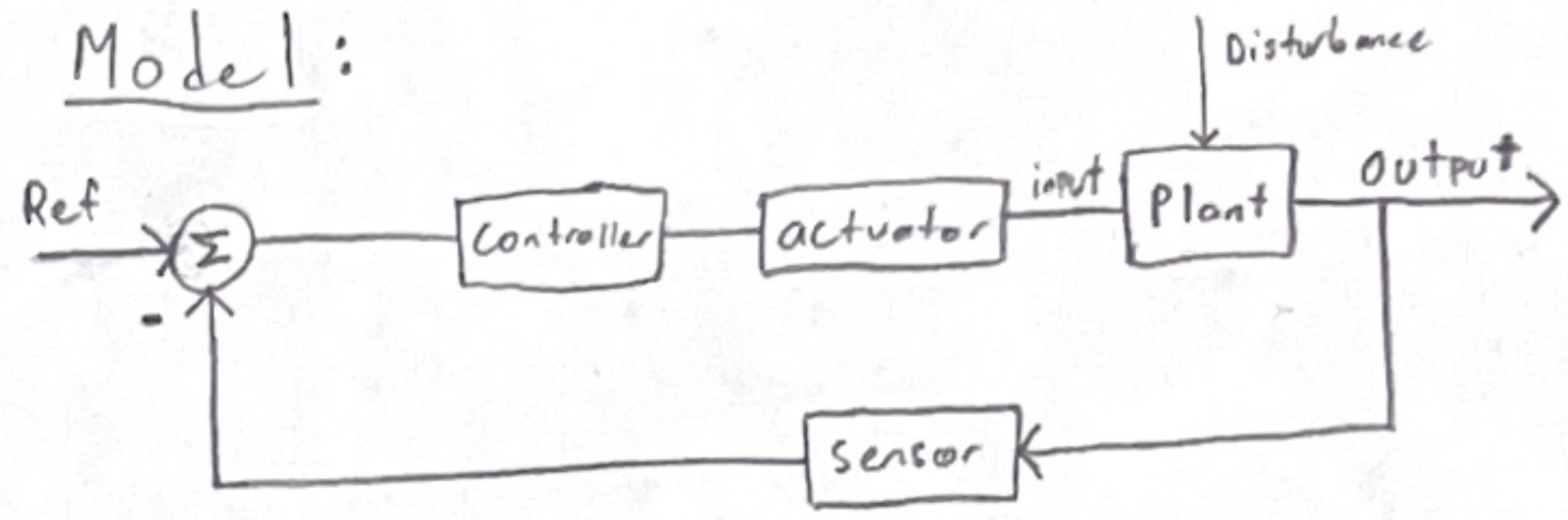
④ Initial Value Thrm, "if the limit exists. ie: no requirements"

$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s)$

⑤ Frequency shift,

$e^{-at}f(t) \longleftrightarrow F(s+a)$

Model:

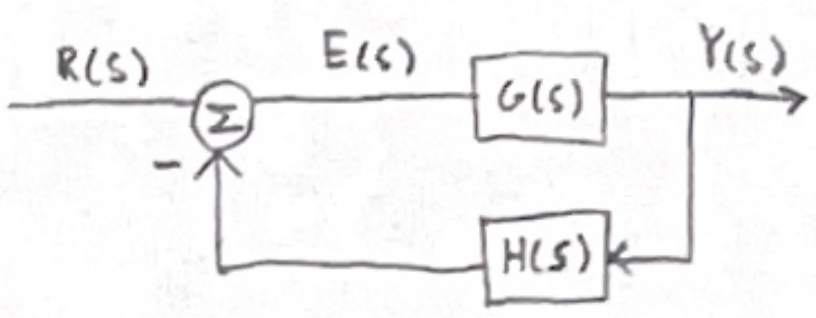


Similarities:

① Capacitor ~ Mass ~ Inertia

② Inductor ~ Spring ~ Rotational spring

③ Resistor ~ Damper ~ Friction



Definitions:

① G(s) = Forward Transfer Function

② H(s) = Feedback Transfer Function

③ G(s)H(s) = Open-Loop TF

④ $\frac{C(s)}{R(s)}$ = Closed-loop TF

⑤ $\frac{C(s)}{E(s)}$ = Feed-forward TF

Black's Formula:

$Y(s) = \frac{G(s)}{1+G(s)H(s)} \cdot R(s)$

*if Positive Feedback, ⊕ → ⊖

State-Space Modeling:

$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = [A]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [B][u]$

$[y] = [C]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + [D][u]$

$T(s) = C\cdot(sI-A)^{-1}\cdot B + D$

Electrical Elements:

Resistor: $Z = R$

Inductor: $Z = sL$

Capacitor: $Z = \frac{1}{sC}$

Stability:

① STABLE → All poles in LHP

② Marginally STABLE
- No poles in RHP
- All poles on $j\omega$ axis have multiplicity 1

③ UNSTABLE → Poles in RHP

Tricks:

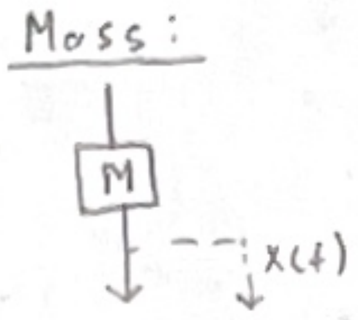
① 1st Order, all coefficients have the same sign then all poles in LHP

② 2nd order, all coefficients have the same sign then all poles in LHP

③ 3rd order, if even one coefficient has a different sign you will have roots in RHP

Mechanical Elements:

Mass:

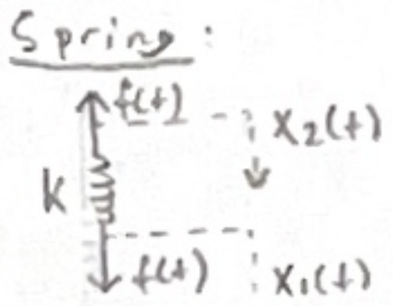


$f(t) = Mx''(t)$

$\updownarrow$

$F(s) = Ms^2X(s)$

Spring:

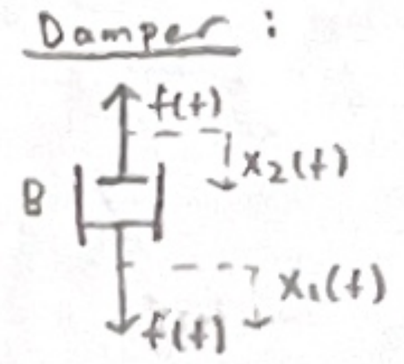


$f(t) = k[x_1(t) - x_2(t)]$

$\updownarrow$

$F(s) = k[X_1(s) - X_2(s)]$

Damper:



$f(t) = B[x_1'(t) - x_2'(t)]$

$\updownarrow$

$F(s) = Bs[X_1(s) - X_2(s)]$

Routh-Array:

s^4	a_n	a_{n-2}	a_{n-4}
s^3	a_{n-1}	a_{n-3}	
s^2	b_1	b_2	
s^1	c_1		
s^0	d_1		

$b_1 = \frac{(a_{n-1})(a_{n-2}) - (a_n)(a_{n-3})}{(a_{n-1})}$

$b_2 = \frac{(a_{n-1})(a_{n-4}) - (a_n)(a_{n-5})}{(a_{n-1})}$

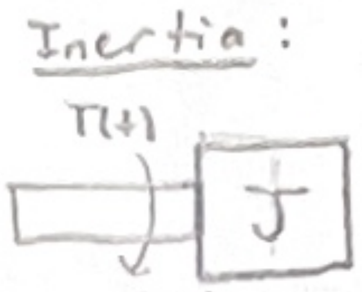
$c_1 = \frac{(b_1)(a_{n-3}) - (a_{n-1})(b_2)}{(b_1)}$

$c_2 = \frac{(b_1)(a_{n-5}) - (a_{n-1})(b_3)}{(b_1)}$

"The number of sign changes in the first column is equal to the number of roots in the RHP"

Rotational Elements:

Inertia:

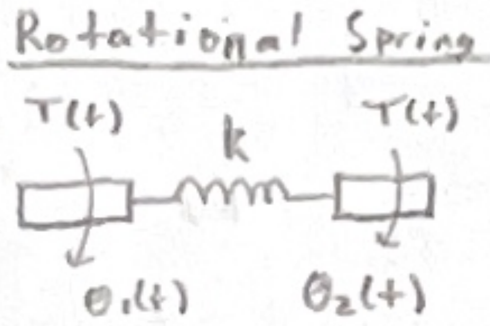


$T(t) = J\theta''(t)$

$\updownarrow$

$T(s) = Js^2\theta(s)$

Rotational Spring:

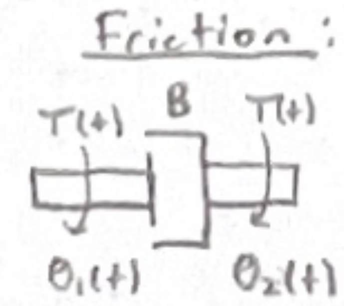


$T(t) = k[\theta_1(t) - \theta_2(t)]$

$\updownarrow$

$T(s) = k[\theta_1(s) - \theta_2(s)]$

Friction:



$T(t) = B[\theta_1'(t) - \theta_2'(t)]$

$\updownarrow$

$T(s) = Bs[\theta_1(s) - \theta_2(s)]$

Ex: Inverse of a 2x2

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Ex: Inverse of a 3x3 Matrix

$A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ $\det(A) = 3$

$\begin{bmatrix} +\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} \end{bmatrix}$

$= \begin{bmatrix} -7 & -5 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ * Flip over Diagonal

Time Response

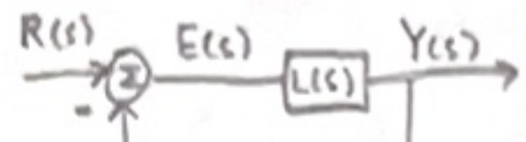
$y(t) = y_t(t) + y_{ss}(t)$ (Transient, Steady-state)

① Transient = 0 as $t \to \infty$

② Steady-state = $\lim_{t\to\infty} y(t)$

Steady-state Error:

$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s \cdot \frac{1}{1+L(s)} \cdot R(s)$



*unity feedback = 1 on the bottom

Error Constants:

Step-Error: $K_p = \lim_{s\to 0} L(s)$

Ramp-Error: $K_v = \lim_{s\to 0} s \cdot L(s)$

Parabolic-Error: $K_a = \lim_{s\to 0} s^2 \cdot L(s)$

ess for Inputs:

① $r(t) = Ru(t)$, $e_{ss} = \frac{R}{1+K_p}$

② $r(t) = Rtu(t)$, $e_{ss} = \frac{R}{K_v}$

③ $r(t) = \frac{Rt^2}{2} u(t)$, $e_{ss} = \frac{R}{K_a}$

Damping Ratios:

① Undamped, $z = 0$

② Underdamped, $0 < z < 1$

③ Critically Damped, $z = 1$

④ Overdamped, $z > 1$

Accurate Tracking ($e_{ss} = 0$):

$K_p = K_v = K_a = \infty$

Time Values Graphically:

① Delay time, time to reach $0.5 y_{ss}$

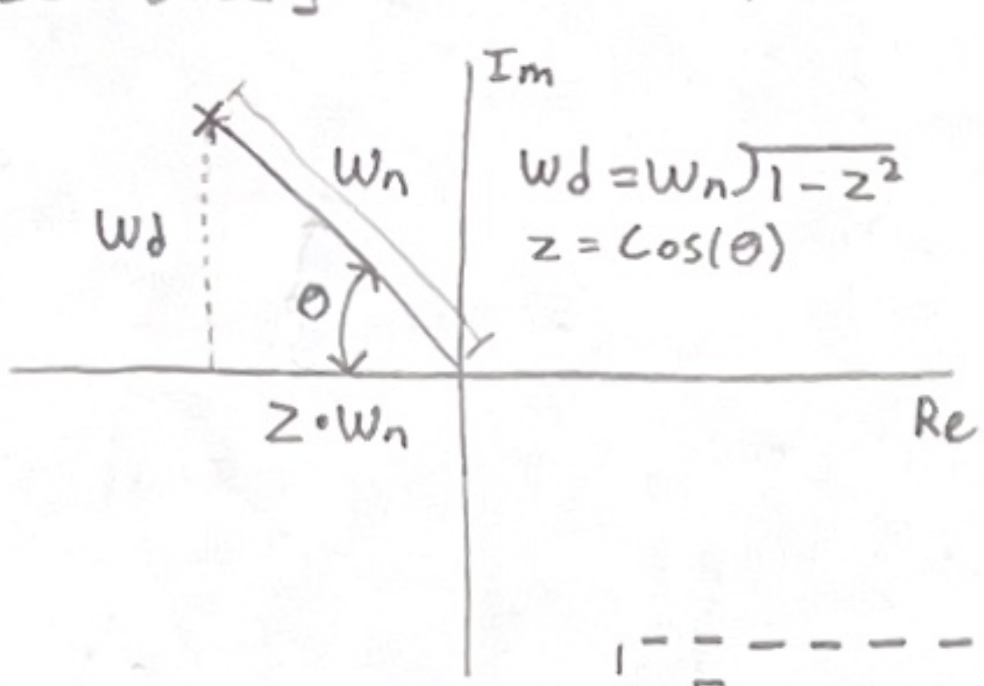
② Rise time, $0.9y_{ss} - 0.1y_{ss}$

③ Settling time, Time when it enters percent box and stays in it

④ Percent overshoot, $\frac{y_{max} - y_{ss}}{y_{ss}} \cdot 100\%$

⑤ Peak Time, time to reach y_{max}

Location of Pole gives you Everything: [2nd order] $*0 < z < 1*$



$W_d = W_n\sqrt{1-z^2}$

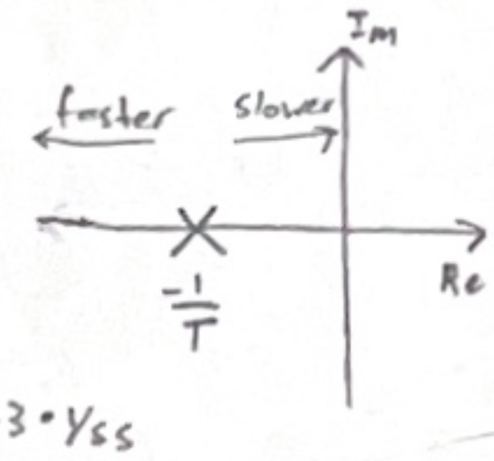
$z = \cos(\theta)$

1st Order System:

$G(s) = \frac{K}{Ts+1}$

① DC Gain, $\lim_{t\to\infty} y(t) = K$

② Time Constant, T when @ $0.63 \cdot y_{ss}$



For Step Response:

① $y_{ss} = K$

② y_{max}, T_p, PO = UNDEFINED

③ Delay time = 0.7T

④ Rise time = 2.2T

⑤ Settling time:

20% = 4T

5% = 3T

2nd Order System:

$G(s) = \frac{K}{s^2 + 2zW_n \cdot s + W_n^2}$

① Damping Ratio, 'z'

② Undamped Natural Frequency, 'W_n'

Properties of 2nd order Systems:

① Settling time, 20% $= \frac{4}{z \cdot W_n}$, 5% $= \frac{3}{z \cdot W_n}$ * $T = \frac{1}{z \cdot W_n}$

② Peak time, $\frac{\pi}{W_d}$

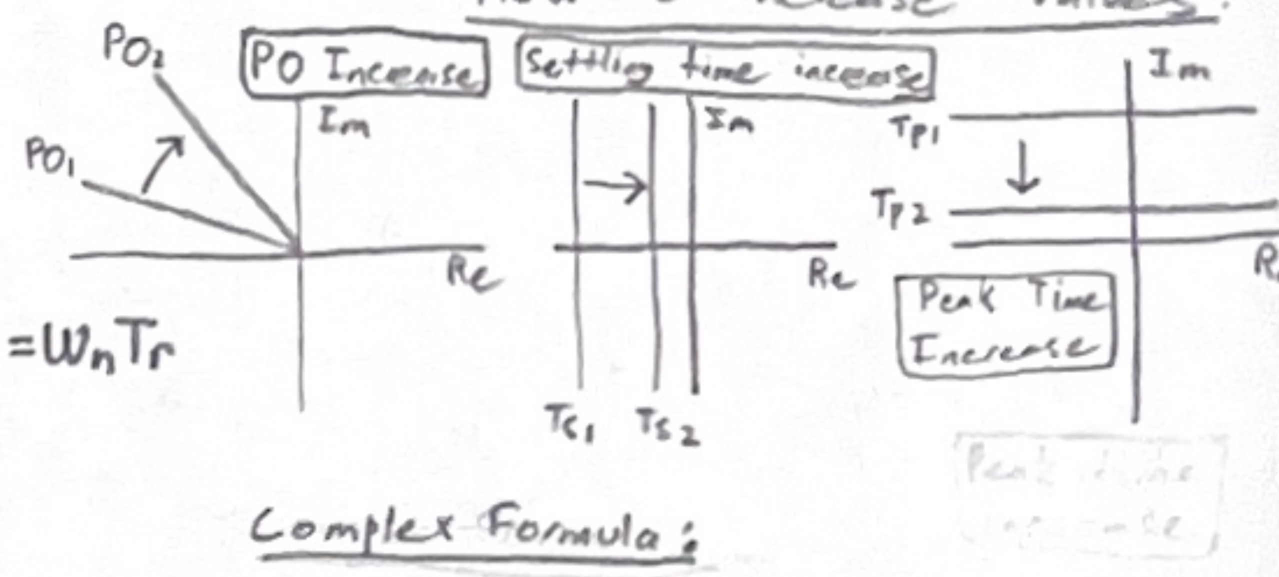
③ Peak Value, $1 + e^{-z\cdot\pi/\sqrt{1-z^2}}$

④ Percent Overshoot, $100e^{-z\cdot\pi/\sqrt{1-z^2}}$

⑤ Rise Time, $1.76z^3 - 0.417z^2 + 1.039z + 1 = W_n T_r$

⑥ y_{max}, $1 + e^{\frac{-z\cdot\pi}{\sqrt{1-z^2}}}$

How to increase Values:



Pole Locations:

① Undamped, Poles on jw axis

② Underdamped, Distinct Complex Poles

③ Critically damped, Double Real Poles

④ Overdamped, Distinct Real Poles

$PO = 100e^{-\pi/\tan\theta}$

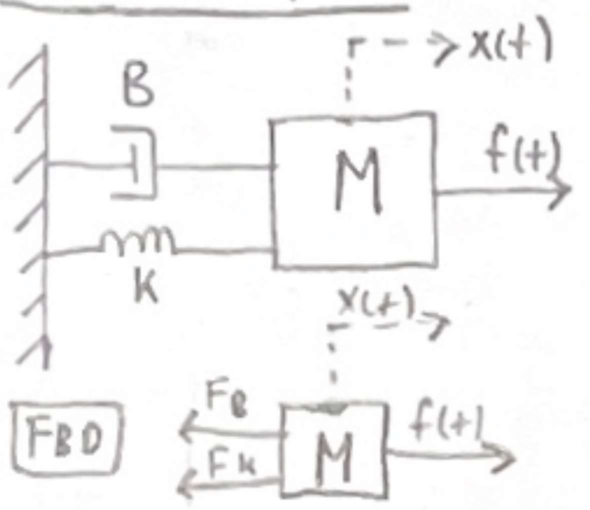
Dominant Poles

$G(s) = \frac{10}{(s+1)(s+10)} \approx \frac{1}{s+1}$ (s-10x Far away)

Complex Formula:

$$z = \frac{\ln(\frac{PO}{100})}{\sqrt{\pi^2 + (\ln(\frac{PO}{100}))^2}}$$

FBD Example:



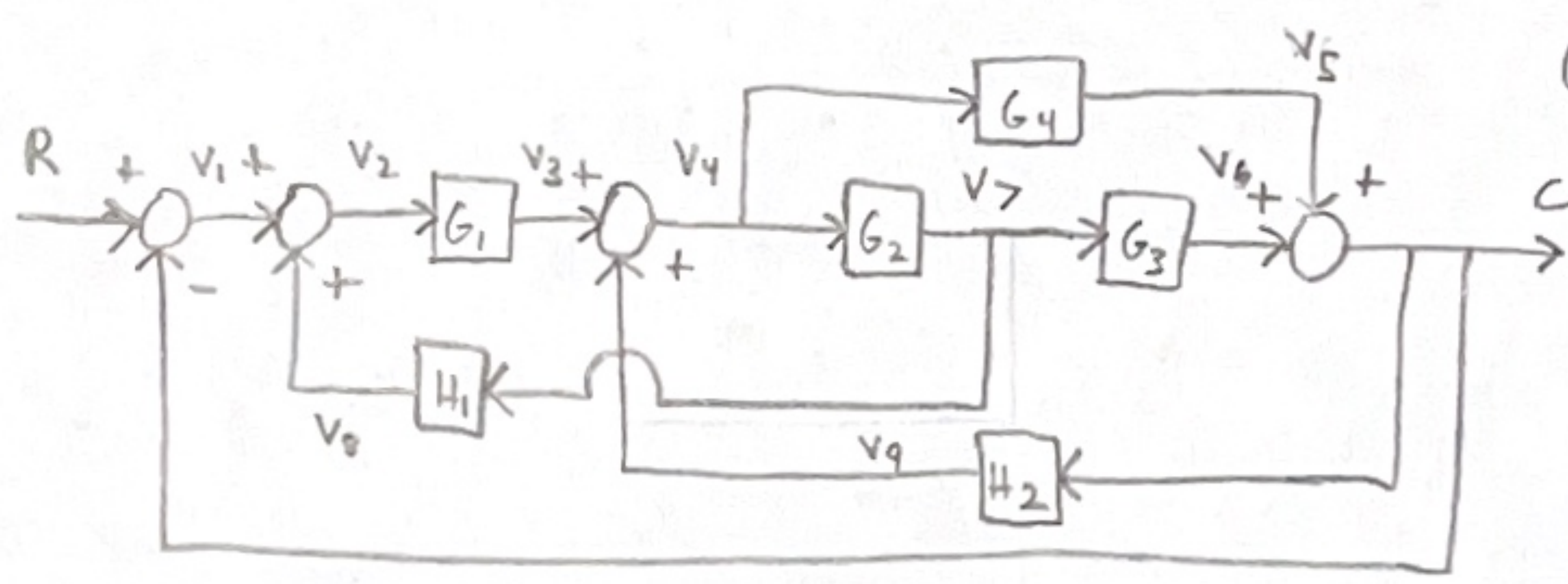
$F_B = (-B)(\dot{X}_{LEFT} - \dot{X}_{RIGHT})$

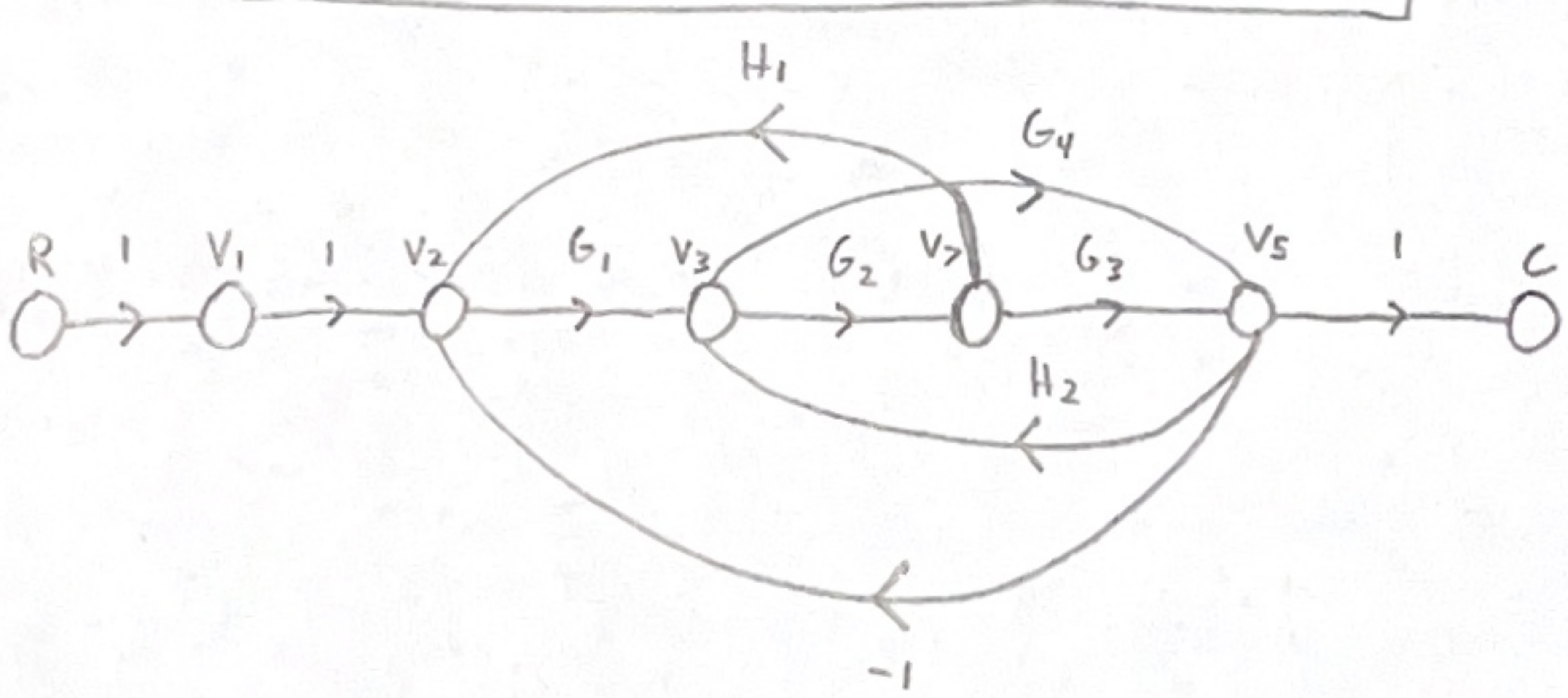
$F_K = (-K)(X_{LEFT} - X_{RIGHT})$

$M\ddot{x} = f(t) - F_B - F_K$

$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K}$

Signal Flow Example:





Ex: "Linearize"

$m\dot{v}(t) + k_f v(t) + k_a v^2(t) = F_m(t)$ @ OP → $V_0 = 30$

① Input: $v(t)$ output: $F_m(t)$

② Operating Point: $V_0 = 30 \to V(t) = V_0 \to \dot{V}(t) = 0$

$\therefore m\dot{v}(t) + k_f v(t) + k_a v^2(t) = F_m(t)$

$K_f V_0 + K_a V_0^2 = F_{M0}$

③ Taylor Series Expansion:

$f(\dot{V}, V, F_M) = m\dot{v}(t) + k_f v(t) + k_a v^2(t) - F_m(t)$

$\frac{df}{d\dot{v}}\Big|_{op} = m$, $\frac{df}{dv}\Big|_{op} = K_f + 2K_a V_0$, $\frac{df}{dF_M}\Big|_{op} = -1$

$\therefore f(\dot{V}, V, F_M) = f(\dot{V}_0, V_0, F_{M0}) + m\delta\dot{V} + (K_f + 2K_a V_0)\delta V - \delta F_M$

$0 = 0 + m\delta\dot{V} + (K_f + 2K_a V_0)\delta V - \delta F_M$

$\therefore \delta F_M = m\delta\dot{V} + (K_f + 2K_a V_0)\delta V$

$F_M - F_{M0} = m(\dot{V} - \dot{V}_0) + (K_f + 2K_a V_0)(V - V_0)$

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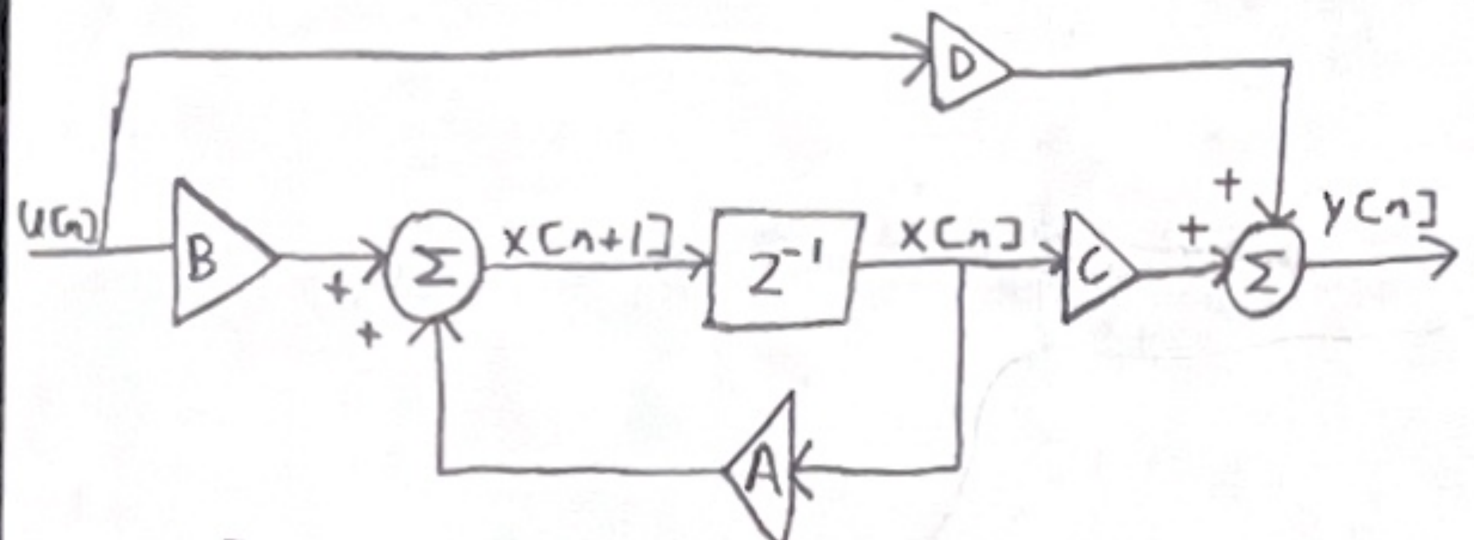
DT System State-Space LTI Model



$x[n+1] = Ax[n] + Bu[n]$

$y[n] = Cx[n] + Du[n]$

State Transition Matrix

$x[n] = \underbrace{A^n x_0}_{ZIR} + \underbrace{\sum_{k=0}^{n-1} A^{n-1-k} Bu[k]}_{ZSR}$

$y[n] = \underbrace{CA^n x_0}_{ZIR} + \underbrace{\sum_{k=0}^{n-1} CA^{n-1-k} Bu[k] + Du[n]}_{ZSR}$

$Y(z) = C(zI-A)^{-1}x_0 + \underbrace{[C(zI-A)^{-1}B + D]}_{H(z)}U(z)$

How to compute State Transition Matrix?

① $\det(A-\lambda I) = 0 \rightarrow Av_i = \lambda_i v_i$

$A = [v_1, v_2 \dots v_n] \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & & \lambda_n \end{bmatrix} [v_1, v_2, \dots, v_n]^{-1} = TDT^{-1}$

State Transition matrix:

CT → $e^{At} = T \begin{bmatrix} e^{\lambda_1 t} & 0 & & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & & e^{\lambda_N t} \end{bmatrix} T^{-1}$

DT → $A^n = T \begin{bmatrix} \lambda_1^n & 0 & & 0 \\ 0 & \lambda_2^n & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & & \lambda_N^n \end{bmatrix} T^{-1}$

② Caley-Hamilton Thereom

State Transition matrix:

CT → $e^{At} = \sum_{k=0}^{N-1} c_k A^k$ where $\forall i \in [1, N]$

$e^{\lambda_i t} = \sum_{k=0}^{n-1} c_k \lambda_i^k$

DT → $A^n = \sum_{k=0}^{N-1} c_k A^k$ where $\forall i \in [1, N]$

$\lambda_i^n = \sum_{k=0}^{N-1} c_k \lambda_i^k$

Frequency Response

$x(t) = e^{j\omega t}$ → [CT] → $y(t) = |H(\omega)| e^{j(\omega t + \angle H(\omega))}$

$x[n] = e^{j\omega n}$ → [DT] → $y[n] = |H(\omega)| e^{j(\omega n + \angle H(\omega))}$

Ex: "Matrix Multiplication"

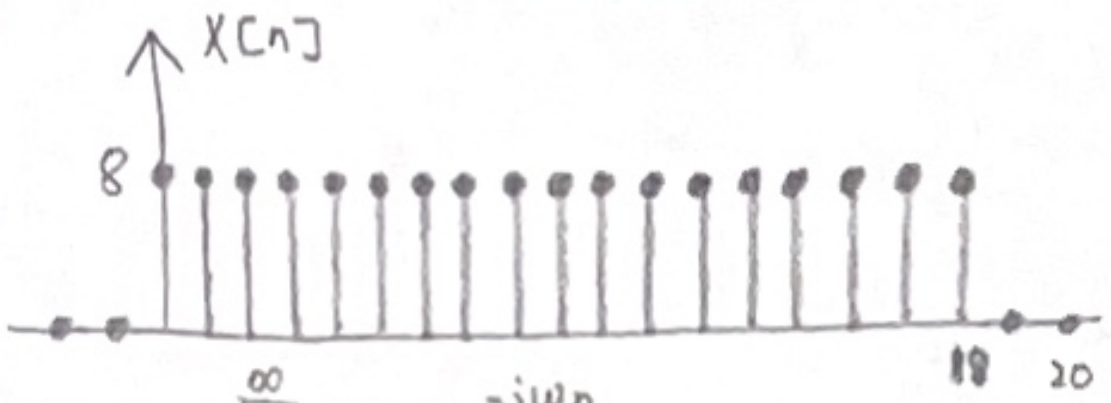
$A = \begin{bmatrix} 3 & -1 & 0 \\ 2 & 5 & 1 \\ -7 & 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 6 & -1 & 0 \\ 0 & 1 & -2 \\ 3 & -8 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} (3)(6)+(-1)(0)+(0)(3) & (3)(-1)+(-1)(1)+(0)(-8) & \dots \\ (2)(6)+(5)(0)+(1)(3) & (2)(-1)+(5)(1)+(1)(-8) & \dots \\ (-7)(6)+(1)(0)+(3)(3) & (-7)(-1)+(1)(1)+(3)(-8) & \dots \end{bmatrix}$

Inverse of a 2x2 Matrix

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad A^{-1} = \frac{1}{\det(A)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Ex: "DT Fourier Transform"



$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$= \sum_{n=0}^{18} (8) e^{-j\omega n} \longrightarrow \frac{8(1-e^{-j\omega n(19)})}{1-e^{-j\omega n}}$

$= \frac{8e^{-j\omega n(19/2)} [e^{j\omega n(19/2)} - e^{-j\omega n(19/2)}]}{e^{-j\omega(1/2)} [e^{j\omega(1/2)} - e^{-j\omega(1/2)}]}$

$= 8e^{-j\omega(9)} \cdot \frac{\sin(19\omega/2)}{\sin(\omega/2)}$

Ex: "Inverse of a 3x3 Matrix"

$A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & -1 & 3 \\ 1 & 1 & 4 \end{bmatrix} \quad \det(A) = 3$

↓

$\begin{bmatrix} +\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 0 & 1 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} \end{bmatrix}$

$= \begin{bmatrix} -7 & -5 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ * Flip over diagonal

$A^{-1} = \frac{1}{3} \begin{bmatrix} -7 & 1 & 1 \\ -5 & -1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$

Ex: "Cayley Hamilton Thereom"

$A = \begin{bmatrix} 0 & 7 \\ -\frac{72}{7} & 17 \end{bmatrix} \rightarrow \lambda^2 - 17\lambda + 72 = 0$

$(\lambda-9)(\lambda-8) \therefore \lambda_1 = 8$, $\lambda_2 = 9$

CT: $e^{\lambda_1 t} = \alpha_0 + (\lambda_1)\alpha_1$

$e^{\lambda_2 t} = \alpha_0 + (\lambda_2)\alpha_1$

DT: $\lambda_1^n = \alpha_0 + (\lambda_1)\alpha_1$

$\lambda_2^n = \alpha_0 + (\lambda_2)\alpha_1$

$e^{At} = A^n = \alpha_0 I + \alpha_1 A$

$f(A) = \alpha_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha_1 \begin{bmatrix} 0 & 7 \\ -\frac{72}{7} & 17 \end{bmatrix}$

For Bigger Matrices:

$e^{At} = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$

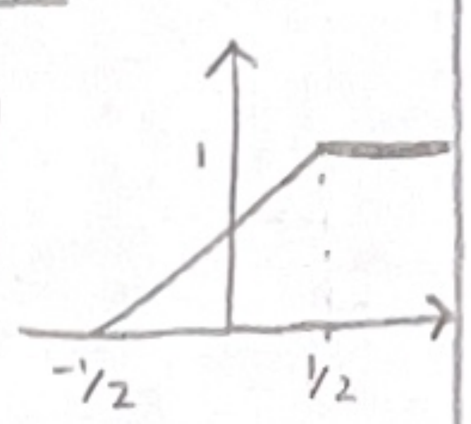
$A^n = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$

Transfer function from State Diagram

$H(s) = \frac{b_0 s^N + b_1 s^{N-1} \dots + b_N}{s^N + a_1 s^{N-1} \dots + a_n}$

Ex: "Differentiation Property"

$x(t) = \begin{cases} 0, & t < -1/2 \\ t + \frac{1}{2}, & -\frac{1}{2} < t < \frac{1}{2} \\ 1, & t > 1/2 \end{cases}$



$x(t) = r(t+\frac{1}{2}) - r(t-\frac{1}{2})$

$\frac{dx(t)}{dt} = u(t+\frac{1}{2}) - u(t-\frac{1}{2})$

$\frac{d^2x(t)}{dt^2} = \delta(t+\frac{1}{2}) - \delta(t-\frac{1}{2})$

↕ FT

$(j\omega)^2 X(\omega) = e^{j(\frac{\omega}{2})} - e^{-j(\frac{\omega}{2})}$

$X(\omega) = \frac{2j\sin(\omega/2)}{(j\omega)^2}$

$X(\omega) = \frac{2\sin(\omega/2)}{j\omega^2}$

Ex: "DT Response"

"Consider a filter with difference equation $y[n] - \frac{1}{4}y[n-2] = x[n] + \frac{1}{2}x[n-1]$"

Find

① Frequency and impulse response

② Output when input is $x[n] = \cos(\frac{\pi n}{2})$

SOLUTION:

$Y(z) - \frac{1}{4}z^{-2}Y(z) = X(z) + \frac{1}{2}z^{-1}X(z)$

$Y(z)[1-\frac{1}{4}z^{-2}] = X(z)[1+\frac{1}{2}z^{-1}]$

$H(z) = \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-2}} = \frac{1}{1-\frac{1}{2}z^{-1}}$

FREQ Response

$H(\omega) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$

IMPULSE RESPONSE:

$H[n] = Z^{-1}\{H(z)\} = (\frac{1}{2})^n u[n]$

$H(\frac{\pi}{2}) = \frac{1}{1-\frac{1}{2}e^{-j(\pi/2)}} = \frac{1}{1+\frac{1}{2}j}$ ← z_1, ← z_2

$|H(\frac{\pi}{2})| = \frac{1}{\sqrt{1+(\frac{1}{2})^2}} \quad \angle H(\frac{\pi}{2}) = 0 - \tan^{-1}(\frac{1/2}{1})$

DT Causalities

① Causal, attach $u[n]$ ie: $(A[n]+B[n])u[n]$

② Two-sided, attach both ie: $A[n]u[n] + B[n](-u[-n-1])$

③ Anti-Causal, attach $(-u[-n-1])$ ie: $(A[n]+B[n])(-u[-n-1])$

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Ex: "Discrete Time Convolution"

"The impulse response of a discrete LTI system is given by $h[n] = u[n] - u[n-5]$. Given that the input to the system is given by $x[n] = 3(u[n] - u[n-6])$. Find $y[n]$.

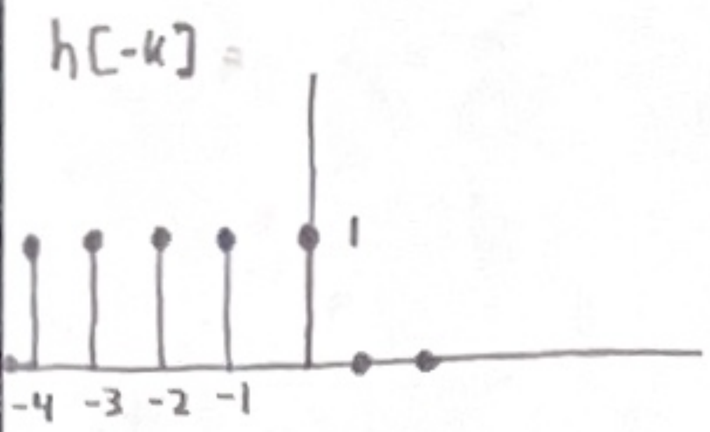
SOLUTION:

$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$h[n] = u[n] - u[n-5]$

Steps:
1. For (+) values of n, shift h[-k] to the right
2. For (-) values shift left
3. For n=0, use h[-k]
4. Sum over all k; x[k]•h[-k+n] and that is the value for y[n]

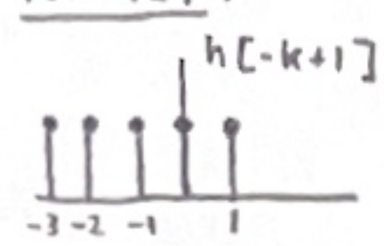
$h[-k]$



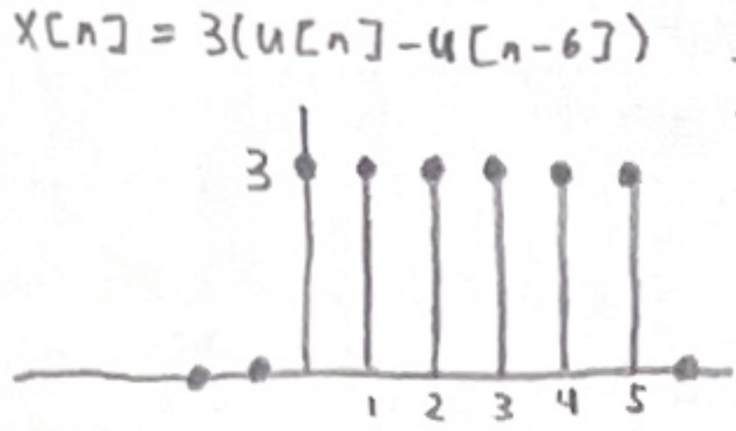
For n=0:
$y[0] = (3)(1) = 3$

For n=1:

h[-k+1]

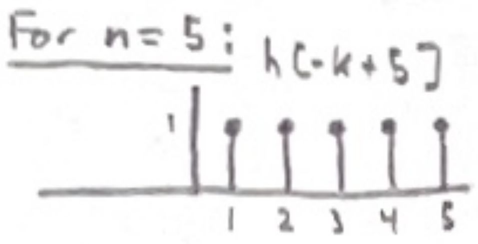


$x[n] = 3(u[n] - u[n-6])$



$y[1] = (3)(1) + (3)(1) = 6$

For n=5: h[-k+5]



$y[5] = (3)(1)+(3)(1)+(3)(1)+(3)(1)+(3)(1) = 15$

Ex: "Aliasing"

"A continuous-time signal $x(t) = 7\cos(23\pi t) + 7\sin(24\pi t) + 4\cos(46\pi t + \pi/9)$ is sampled at 24Hz. Determine w(t) reconstructed using an ideal interpolator at a sampling rate of 1/24 s

SOLUTION:

$f_s = 24$
$W_s = 48\pi$
$T_s = 1/24$

1. $7\cos(23\pi t)$: $2W_m = 46\pi < 48\pi$ ∴ Reconstructed completely
2. $7\sin(24\pi t)$: $2W_m = 48\pi = 48\pi$ ∴ DC value because sine
3. $4\cos(46\pi t + \pi/9)$: $2W_m = 92\pi > 48\pi$ ∴ Aliasing

$4\cos(46\pi t - kt(2\pi(1/T_s)) + \pi/9)$

"Choose integer value for k that puts it within nyquist range ie: 1"

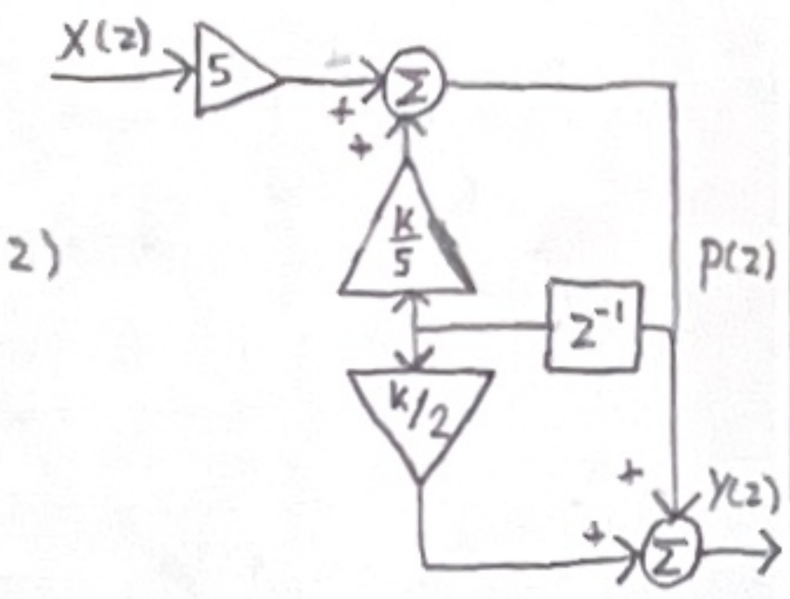
$4\cos(46\pi t - 48\pi t + \pi/9)$

$= 4\cos(-2\pi t + \pi/9)$

$\therefore W(t) = 7\cos(23\pi t) + 4\cos(-2\pi t + \pi/9)$

Ex: "Block Diagram"

1. Find the transfer function H(z)
2. State the Radius of Convergence
3. Find the values of |k| for which the system is BIBO Stable



SOLUTION:

1. $p(z) = 5X(z) + \frac{k}{5}z^{-1}p(z)$

$\therefore p(z) = \frac{5X(z)}{1 - \frac{k}{5}z^{-1}}$

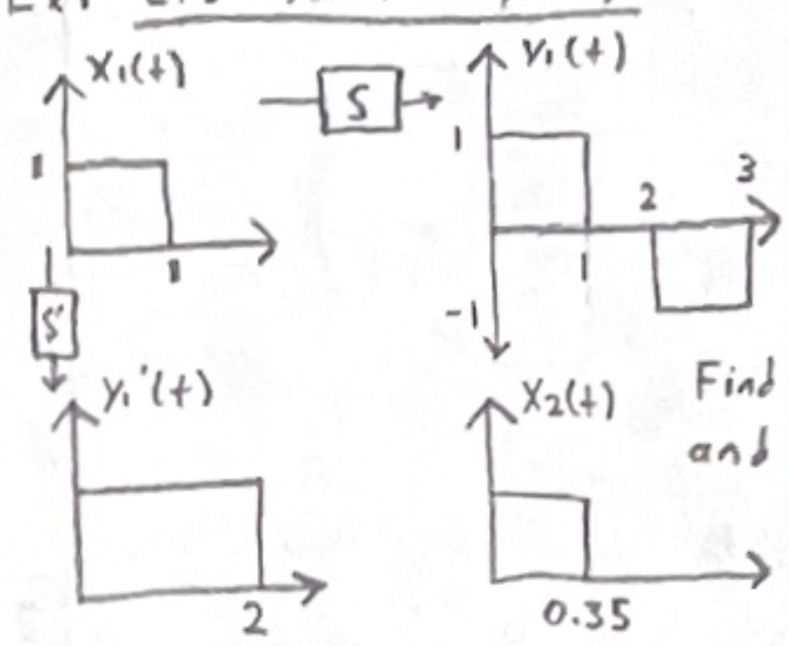
$Y(z) = p(z) + \frac{k}{2}z^{-1}p(z)$

$Y(z) = \frac{5X(z)}{1-\frac{k}{5}z^{-1}} + \frac{k}{2} \cdot \frac{5X(z)z^{-1}}{1-\frac{k}{5}z^{-1}}$

$\frac{Y(z)}{X(z)} = \frac{5}{1-\frac{k}{5}z^{-1}} + \frac{k \cdot 5 \cdot z^{-1}}{2(1-\frac{k}{5}z^{-1})}$ → $H(z) = \frac{5(z+\frac{k}{2})}{z - \frac{k}{5}}$

2. ROC: $(|\frac{k}{5}|, \infty)$
3. BIBO: (0, 5)

Ex: "LTI System Property"



Find $y_2(t)$ and $y_2'(t)$

SOLUTION:

$x_1(t) = u(t) - u(t-1)$
$y_1(t) = x_1(t) - x_1(t-2)$
$y_1'(t) = x_1(t) + x_1(t-1)$
$h(t) = \delta(t) - \delta(t-2)$
$h'(t) = \delta(t) + \delta(t-1)$

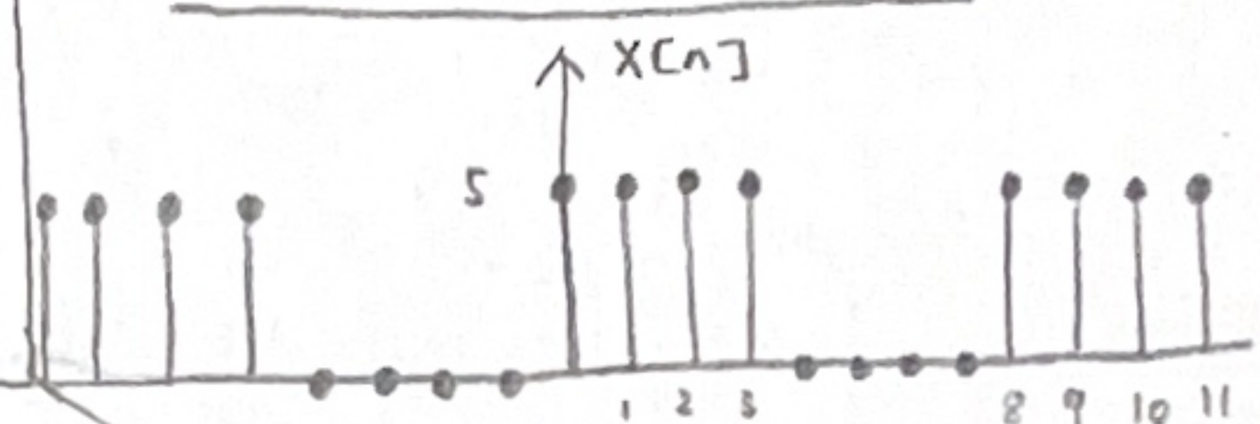
→ $x_2(t) = u(t) - u(t-0.35)$
∴ $y_2(t) = u(t) - u(t-0.35) - u(t-2) + u(t-0.35-2)$
$y_2'(t) = u(t) - u(t-0.35) + u(t-1) - u(t-0.35-1)$

Convolution Properties:
(i) $x(t) * \delta(t) = x(t)$
(ii) $x(t-\alpha) * h(t-\beta) = x(t-\alpha-\beta)$
(iii) $(x_1(t)+x_2(t)) * h(t) = x_1(t)*h(t) + x_2(t)*h(t)$

Ex: "Different Causalities"

Impulse Response:	Laplace Transform:	ROC:	BIBO:
$8e^{-6t}u(t) + 15e^{2t}u(t)$	$\frac{8}{s+6} + \frac{15}{s-2}$	(2, INF)	NO
$-7e^{-4t}u(-t) - 15e^{2t}u(-t)$	$\frac{-7}{(-1)(s+4)} - \frac{15}{(-1)(s-2)}$	(-INF, -4)	NO
$19e^{-6t}u(t) - 13e^{2t}u(-t)$	$\frac{19}{s+6} - \frac{13}{(-1)(s-2)}$	(-6, 2)	YES

Ex: "DT Fourier Series Coefficients"



Find the fourier coefficients X_k for $k \neq 0$.

SOLUTION:

$N = 8$

$W_0 = \frac{2\pi}{N} = \frac{\pi}{4}$

$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(\pi/4)n}$

$X_k = \frac{1}{8}\sum_{n=0}^{3}(5)e^{-jk(\pi/4)n} + \varnothing$

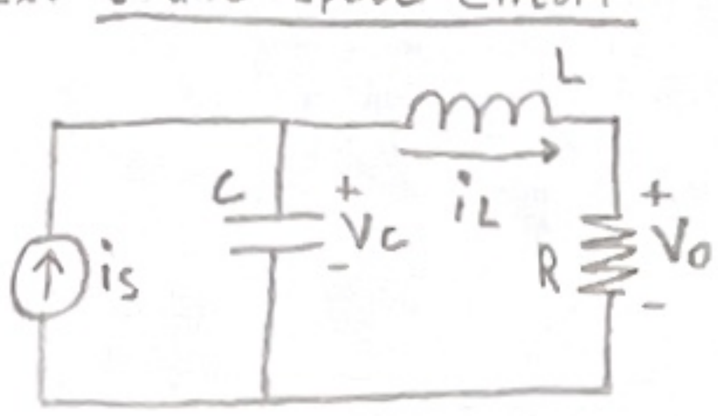
Known:
$a = 5$
$r = e^{-jk(\pi/4)}$
$N = 3$

$\sum_{n=0}^{N} ar^n = \frac{5(1 - e^{-jk(\pi/4)\cdot 4})}{1 - e^{-jk(\pi/4)}}$

$= \frac{5 \cdot e^{-jk(\pi/2)} \cdot [e^{jk(\pi/2)} - e^{-jk(\pi/2)}]}{e^{-jk(\pi/8)} \cdot [e^{jk(\pi/8)} - e^{-jk(\pi/8)}]}$

→ $X_k = \frac{5}{8}e^{-jk(3\pi/8)} \cdot \frac{\sin(k\pi/2)}{\sin(k\pi/8)}$

Ex: "State Space Circuit"



Given:
$R = 77\Omega$
$L = 7H$
$C = 1/196 F$

Known:
$i = C\frac{dV_c}{dt}$
$V = L\frac{di_L}{dt}$

Find:
1. State Space Representation [A, B, C, D]
2. Transfer function H(s)
3. State Transition Matrix Φ(t)
4. Time domain response with zero-input and initial conditions $x_1(0) = 4$, $x_2(0) = 5$

SOLUTION:

KCL: $i_s = C\frac{dV_c}{dt} + i_L$ KVL: $V_c - L\frac{di_L}{dt} - R \cdot i_L = 0$

$C\frac{dV_c}{dt} = i_s - i_L$ $L\frac{di_L}{dt} = V_c - R \cdot i_L$

$\dot{V_c} = \frac{1}{C}i_s - \frac{1}{C}i_L$ $\dot{i_L} = \frac{1}{L}V_c - \frac{R}{L}i_L$

KCL: $\frac{V_o}{R} = i_L$ → $V_o = R \cdot i_L$

$\begin{bmatrix}\dot{V_c} \\ \dot{i_L}\end{bmatrix} = \begin{bmatrix}0 & -1/C \\ 1/L & -R/L\end{bmatrix}\begin{bmatrix}V_c \\ i_L\end{bmatrix} + \begin{bmatrix}1/C \\ 0\end{bmatrix}i_s$ (A, B)

$[V_o] = [0 \; R]\begin{bmatrix}V_c \\ i_L\end{bmatrix} + [0]i_s$ (C, D)

2. KCL: $i_s = V_1(s \cdot C) + \frac{V_1}{s \cdot L + R}$

$\frac{V_o}{i_s} = \frac{R}{CLs^2 + CRs + 1}$

3. $\vec{x} = \begin{bmatrix}V_c \\ i_L\end{bmatrix}$ $A = \begin{bmatrix}0 & -196 \\ 1/7 & -11\end{bmatrix}$

$\det(A - \lambda I) = \begin{vmatrix}0-\lambda & -196 \\ 1/7 & -11-\lambda\end{vmatrix}$

$(-\lambda)(-11-\lambda) + \frac{196}{7} = 0$

$\lambda^2 + 11\lambda + \frac{196}{7} = 0$

$\lambda_1 = -4, \lambda_2 = -7$

For $\lambda = -4$:

$A = \begin{bmatrix}4 & -196 \\ 1/7 & -7\end{bmatrix}\begin{bmatrix}a_1 \\ a_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$

$\frac{1}{7}a_1 - 7a_2 = 0$
$\frac{1}{7}a_1 = 7a_2$
$a_1 = 49a_2$

$\vec{a}^{(1)} = \begin{pmatrix}49 \\ 1\end{pmatrix}$

For $\lambda = -7$:

$A = \begin{bmatrix}7 & -196 \\ 1/7 & -4\end{bmatrix}\begin{bmatrix}a_1 \\ a_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$

$\frac{1}{7}a_1 = 4a_2$
$a_1 = 28a_2$

$\vec{a}^{(2)} = \begin{pmatrix}28 \\ 1\end{pmatrix}$

$e^{At} = \begin{bmatrix}49 & 28 \\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{-4t} & 0 \\ 0 & e^{-7t}\end{bmatrix}\begin{bmatrix}49 & 28 \\ 1 & 1\end{bmatrix}^{-1}$

4. $\begin{bmatrix}X_1(t) \\ X_2(t)\end{bmatrix} = \left(\begin{bmatrix}49 & 28 \\ 1 & 1\end{bmatrix}\begin{bmatrix}e^{-4t} & 0 \\ 0 & e^{-7t}\end{bmatrix}\begin{bmatrix}49 & 28 \\ 1 & 1\end{bmatrix}^{-1}\right)\begin{bmatrix}4 \\ 5\end{bmatrix}$

X_0 (I.C.'s)

Ex: "Diagonalization of a 3x3 Matrix"

$A = \begin{pmatrix}2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 20\end{pmatrix}$ → $\det(A - \lambda I) = \begin{vmatrix}2-\lambda & 1 & 3 \\ 1 & 2-\lambda & 3 \\ 3 & 3 & 20-\lambda\end{vmatrix}$

$= (2-\lambda)[(2-\lambda)(20-\lambda)-9] - (1)[(20-\lambda)-9] + (3)[3-3(2-\lambda)]$

$= \lambda^3 - 24\lambda^2 + 65\lambda - 42$

1. Guess root $(\lambda - 1)$

$= (\lambda-1)(\lambda^2 - 23\lambda + 42)$
$= (\lambda-1)(\lambda-2)(\lambda-21)$

For $\lambda = 2$: *reduce to two variable*

$A = \begin{pmatrix}0 & 1 & 3 \\ 1 & 0 & 3 \\ 3 & 3 & 18\end{pmatrix} \xrightarrow{RREF} \begin{pmatrix}1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0\end{pmatrix}$

→ $\begin{pmatrix}1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a_1 \\ a_2 \\ a_3\end{pmatrix} = \begin{pmatrix}0 \\ 0 \\ 0\end{pmatrix}$

$a_1 = -3a_3$
$a_2 = -3a_3$
$a_3 = a_3$

$\vec{a}^{(1)} = \begin{pmatrix}-3 \\ -3 \\ 1\end{pmatrix}$

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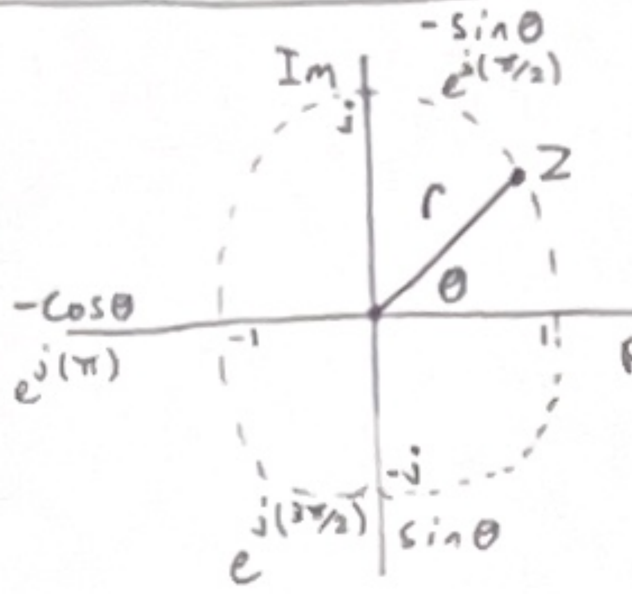
MT1 Material

Classifications of Signals

Complex Numbers:



Cartesian Form: $z = x + jy$

Rectangular Form: $z = (x, y)$

Polar Form: $z = r\angle\theta$

Exponential Form: $z = re^{j\theta}$

Euler Identity: $e^{j\theta} = \cos\theta + j\sin\theta$

① Deterministic: VS. Stochastic:
represented by a formula — Probabilistic Distribution

② Continuous: VS. Discrete:
Domain or range — Domain or range

③ Even: VS. ODD:
$x(t) = x(-t)$ — $x(t) = -x(-t)$

④ Periodic VS. Aperiodic:
$x(t + kT) = x(t)$ — No repetition

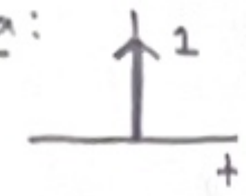
⑤ Finite Ex/Px: VS. Infinite Ex/Px:
$E_x < \infty$, $P_x < \infty$ — $E_x \to \infty$, $P_x \to \infty$

⑥ Causal: VS. Acausal:
$x(t) = 0;\ t < 0$ — Both sides of time (2 sided)

Anticausal: $x(t) = 0,\ t > 0$

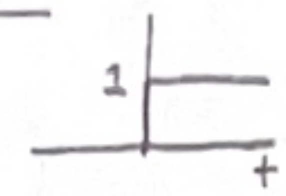
Basic Signals

Dirac Delta: $\delta(t)$

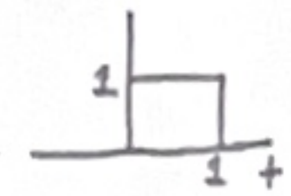


Unit Ramp: $r(t)$ $\begin{cases} t, & t > 0 \\ 0, & t < 0 \end{cases}$

Unit Step: $u(t)$ $\begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$

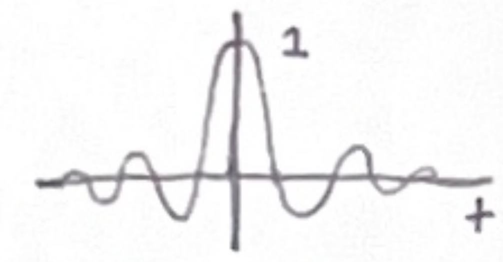


Rect function: $\text{rect}(t)$ $\begin{cases} 1, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$



Impulse Train: $\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

Sinc Function: $\text{Sinc}(t) = \frac{\sin(\pi t)}{\pi t}$



Even/Odd Signals

$y(t) = y_e(t) + y_o(t)$

$y_e(t) = \frac{1}{2}[y(t) + y(-t)]$

$y_o(t) = \frac{1}{2}[y(t) - y(-t)]$

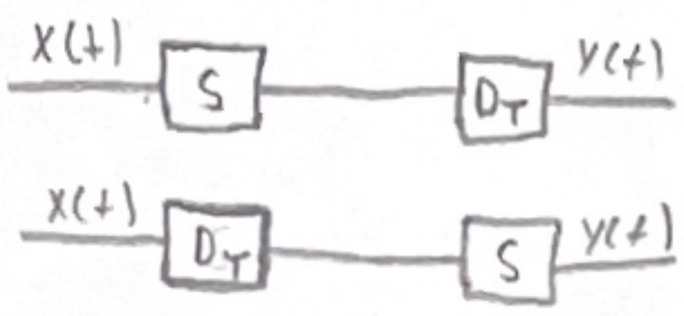
Energy/Power

$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$

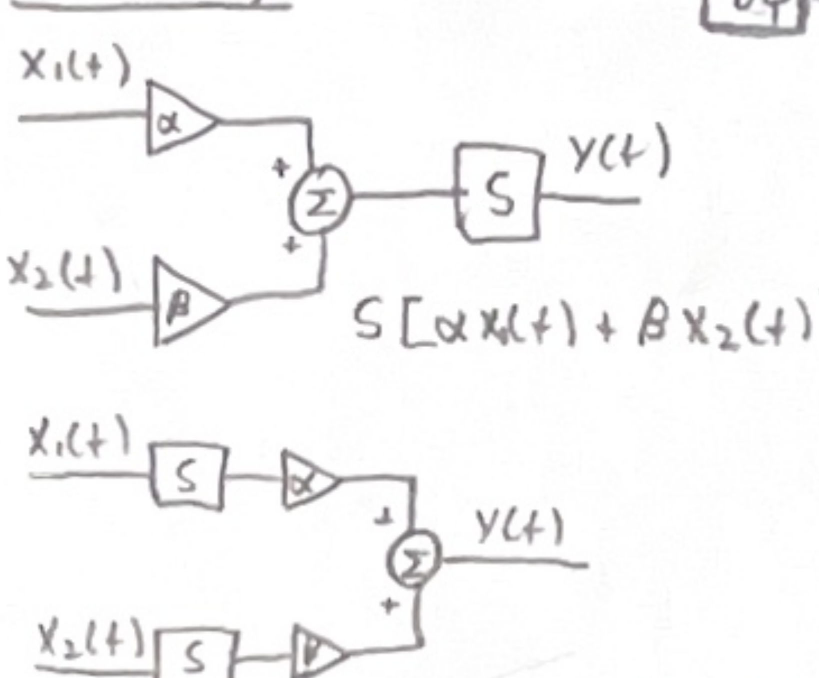
Systems

Time Invariance:



equivilant for Time-Invariance

Linearity:



$S[\alpha x_1(t) + \beta x_2(t)]$

$\alpha S[x_1(t)] + \beta S[x_2(t)]$

equivilant For Linearity

Convolution Integral

$f(t) * g(t) = g(t) * f(t)$

$\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau = \int_{-\infty}^{\infty} g(t-\tau)f(\tau)d\tau$

Causal System

$h(t) = 0,\ t < 0$ "impulse response"

BIBO Stability

$\int_{-\infty}^{\infty} |h(t)|dt < \infty$

Asymptotic Stability

$\text{Re}(\lambda_i) < 0$ "Poles of transfer function"

Frequencies

$\omega = 2\pi f = \frac{2\pi}{T}$

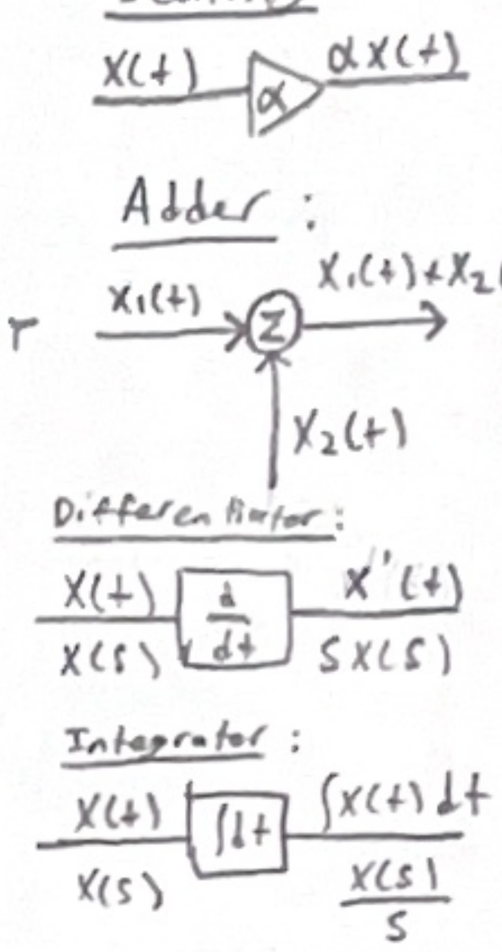
Signal Operations

Scaling: $x(t) \to \alpha \to \alpha x(t)$

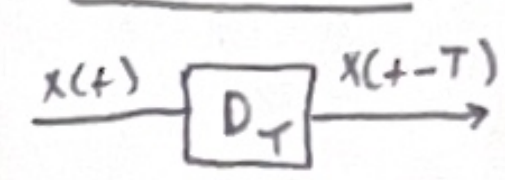
Adder: $x_1(t) + x_2(t)$

Differentiator: $x(t) \to \frac{d}{dt} \to x'(t)$; $X(s) \to sX(s)$

Integrator: $x(t) \to \int dt \to \int x(t)dt$; $X(s) \to \frac{X(s)}{s}$

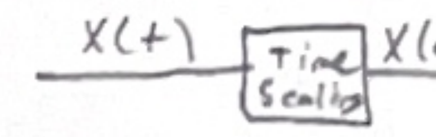


Time Shift: $x(t) \to D_T \to x(t - T)$



$x(t - T)$ DELAY
$x(t + T)$ ADVANCE

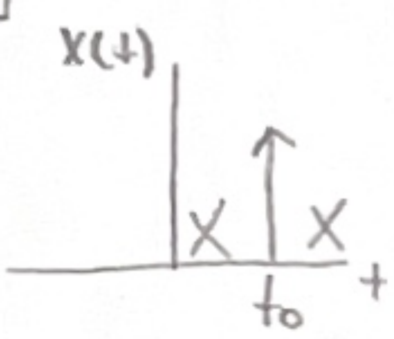
Time Scaling: $x(t) \to$ Time Scaling $\to x(at)$



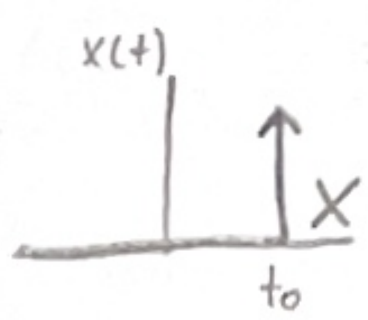
$|a| < 1$ STRETCH
$|a| > 1$ COMPRESS
$a = (-)$ REFLECT

System Properties

① Memoryless: "y(t) at to depends on x(t) at to"



② Causality: "y(t) at to depends on x(t) at to and the past but not future"



Fourier Series

$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$

$X_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t} dt$

Power (FS)

$P_x = \frac{1}{T}\int_T |x(t)|^2 dt$

$P_x = \sum_{k=-\infty}^{\infty} |X_k|^2$

Trigonometric Representations

$x(t) = X_0 + 2\sum_{k=1}^{\infty} |X_k|\cos(k\omega_0 t + \theta_k)$

$= C_0 + 2\sum_{k=1}^{\infty}[c_k\cos(k\omega_0 t) + d_k\sin(k\omega_0 t)]$

$\text{Re}(X_k) = C_k = \frac{1}{T}\int_T x(t)\cos(k\omega_0 t)dt$

$\text{Im}(X_k) = d_k = \frac{1}{T}\int_T x(t)\sin(k\omega_0 t)dt$

$|X_k| = \sqrt{C_k^2 + d_k^2}$

$\theta_k = -\tan^{-1}\left(\frac{d_k}{c_k}\right)$

*Note: $C_0 = X_0$ (DC component)

Frequency Response

EigenFunction Property, if the input is a complex exponential (or sinusoid), then $y_{ss}(t)$ (steady state) is given by: (input $x(t) = \alpha + \cos(\omega_0 t + \theta)$)

① $y_{ss}(t) = |H(j0)|\alpha + |H(j\omega_0)|\cos(\omega t + \theta + \angle H(j\omega_0))$

② $x(t) = e^{j\omega_0 t}$: $y_{ss}(t) = |H(j0)| + |H(j\omega_0)|e^{j(\omega t + \angle H(j\omega_0))}$

MT2 Material

Fourier Transform

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}dt$ IFT

$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ FT

Fourier Transform From Laplace Transform

"If the ROC of $X(s) = L[x(t)]$ contains the $j\omega$ axis, then the FT of $x(t)$ is given by: $X(\omega) = X(s)|_{s=j\omega}$

Fourier Transform from Fourier Series

"If $x(t)$ is a periodic signal with fourier series, $X(\omega) = \sum_k 2\pi X_k \delta(\omega - k\omega_0)$

Energy (FT)

$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$

*Fourier Series from Laplace Transform

$X_k = \frac{1}{T_0} L[x(t)]\Big|_{s = jk\omega_0}$ where: T_0 is the fundamental period

Nyquist

$\omega_s > 2\omega_m$

$f_s = \frac{1}{T_s} = \frac{\omega_s}{2\pi}$

Sampled Signal

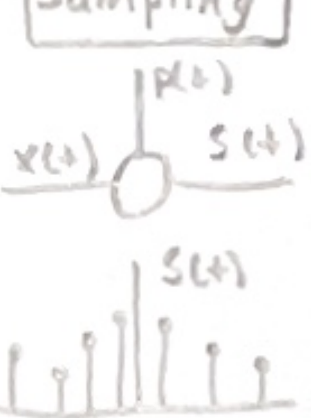
$s(t) = x(t)p(t) \xleftrightarrow{FT} S(\omega) = \frac{1}{2\pi}X(\omega) * P(\omega)$

Sampling

$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$

$\updownarrow$ FT

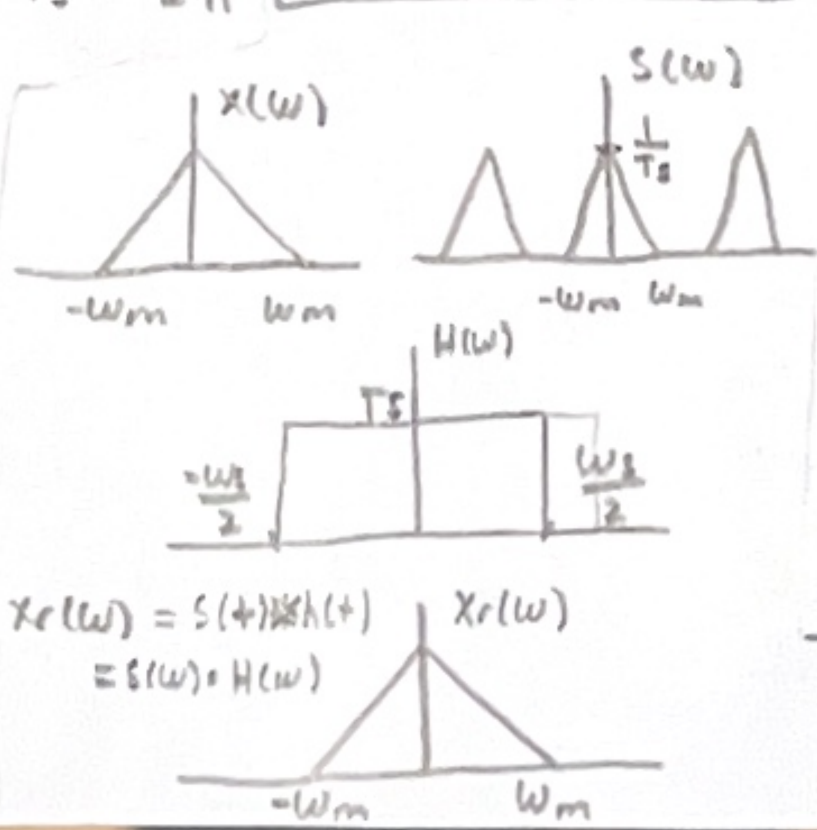
$\frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$

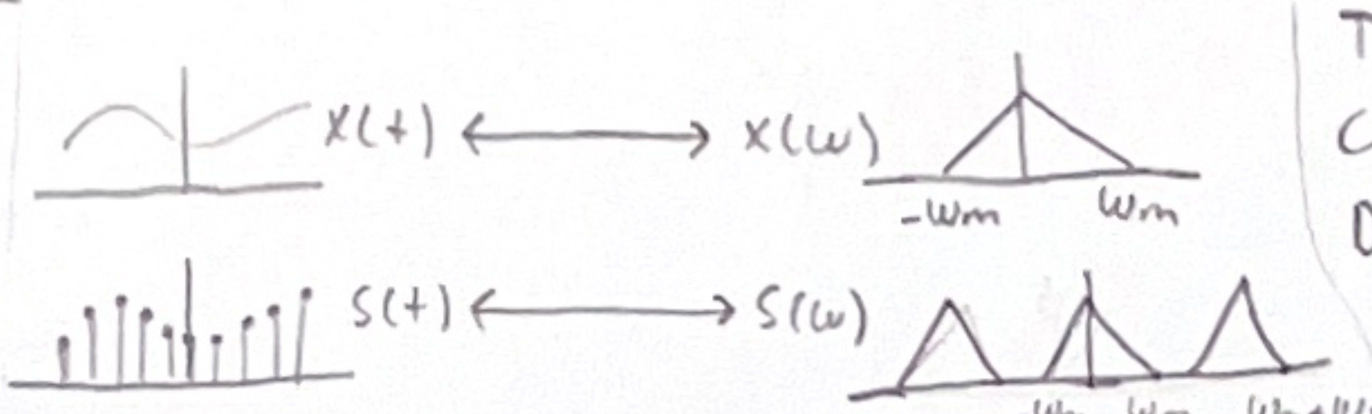


Reconstructing a Signal

$S(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$

$S(\omega) = \frac{1}{T_s}(\ldots X(\omega + \omega_s) + X(\omega) + X(\omega - \omega_s) \ldots)$





Periods/Frequencies

$\omega \to$ GCF (as big as possible)
$T \to$ LCM (as small as possible)
CT → use ω (Frequencies)
DT → use T (Periods, must be a whole number)

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Discrete time signals and systems

Energy / Power:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

BIBO stability:

$$\sum_k |h[k]| < \infty$$

Asymptotic stability:

$|\lambda_i| < 0$ "Poles of transfer function within unit circle"

Convolution Sum:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

Causality

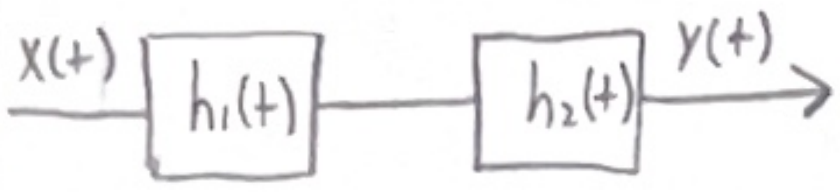
Signal: $x[n]$ is causal if $x[n]=0,\ n<0$

System: The system is causal if $h[n]=0,\ n<0$

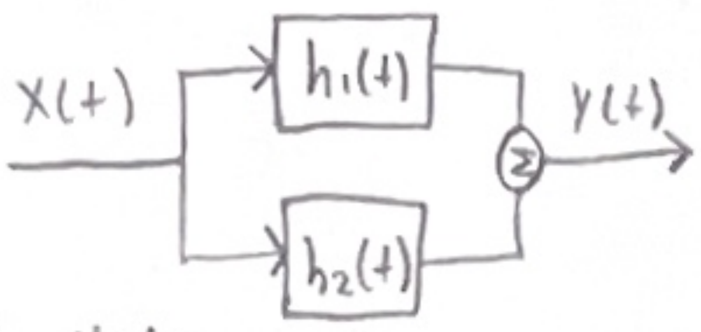
Causal LTI System:

$$y[n] = \sum_{k=0}^{n} x[k]h[n-k],\ n\geq$$

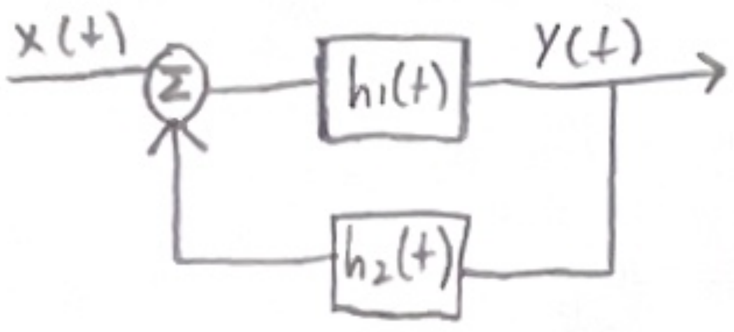
LTI System Connections



Cascade: $H(s) = H_1(s) \cdot H_2(s)$

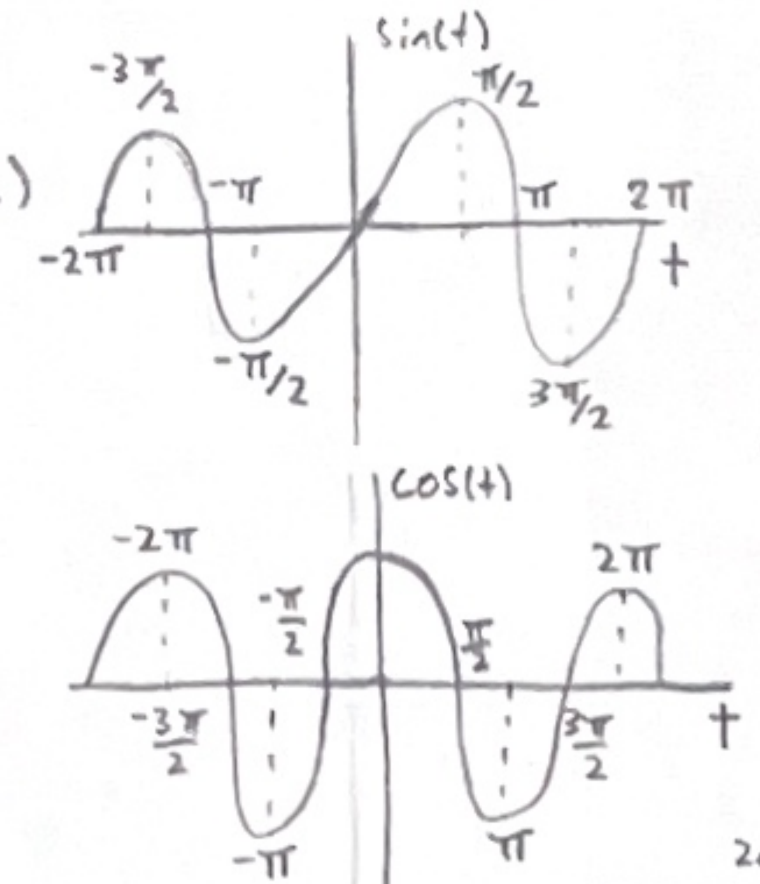


Parallel: $H(s) = H_1(s) + H_2(s)$

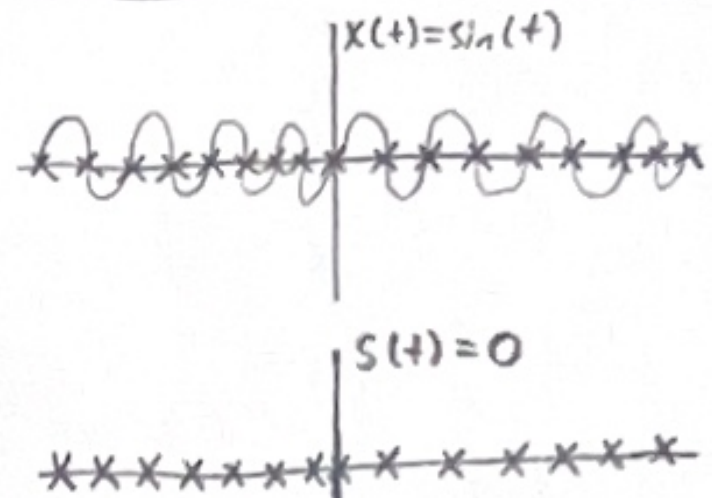


Feedback: $H(s) = \dfrac{H_1(s)}{1 + H_2(s) \cdot H_1(s)}$

Sine / Cosine Graphs

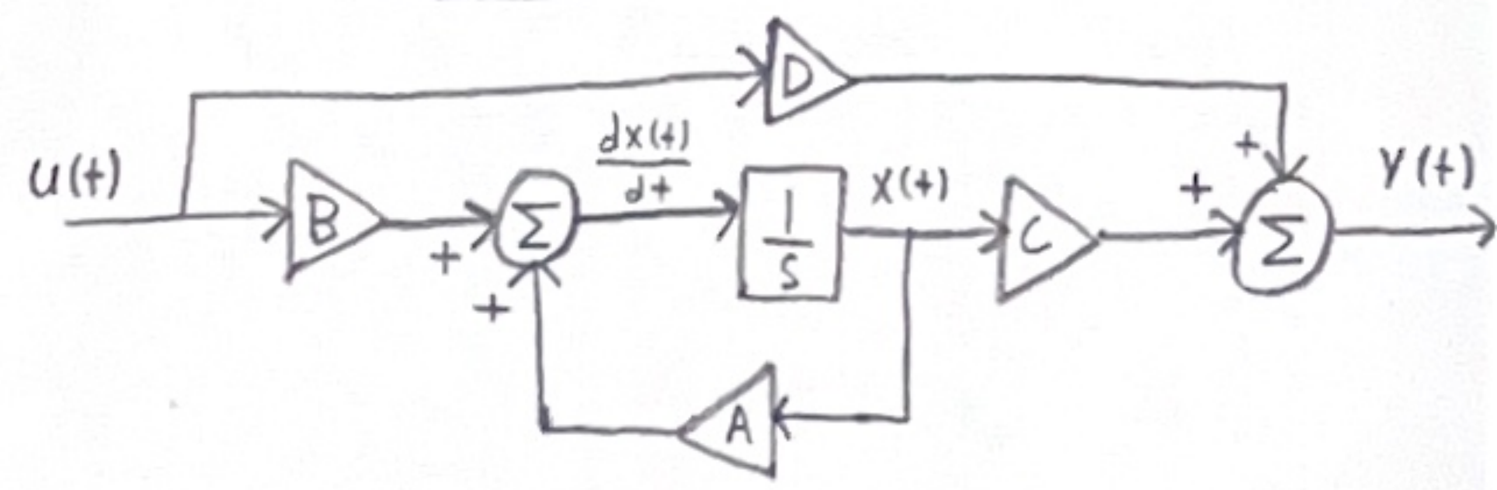


Problem with $\omega_s = 2\omega_m$



"Not aliased but amplitude reduced. Extreme @ sin(t), ok @ cos(t)"

CT System State-Space LTI Model



$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \quad \text{State Differential eqn}$$

(u(t) → input)

output: $y(t) = Cx(t) + Du(t)$ Output eqn

State Transition Matrix (e^{At})

$$y(t) = \underbrace{Ce^{At} \cdot x_0}_{\text{zero-input response}} + \overbrace{C\int_0^t e^{A(t-\tau)} \cdot B \cdot u(\tau)\,d\tau + D \cdot u(\tau)}^{\text{zero-state response}}$$

$\updownarrow \mathcal{L}\{y(t)\}$

$$y(s) = C(sI-A)^{-1}x_0 + [C(sI-A)^{-1} \cdot B + D]\,u(s)$$

$H(s) \to$ T.F Matrix

$$x(t) = \underbrace{e^{At} \cdot x_0}_{\text{zero-input response}} + \underbrace{\int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau}_{\text{zero-state response}}$$

$\updownarrow \mathcal{L}\{x(t)\}$

$$x(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1} \cdot B \cdot u(s)$$

Controllability Matrix

$$x(t_f) = e^{At}x_0 + \int_0^{t_f} e^{A(t_f-\tau)} B\,u(\tau)\,d\tau$$

$$M_c = [B, AB, A^2B, A^{N-1}B]$$

$\det(M_c) \neq 0 \to$ Controllable

Observability Matrix

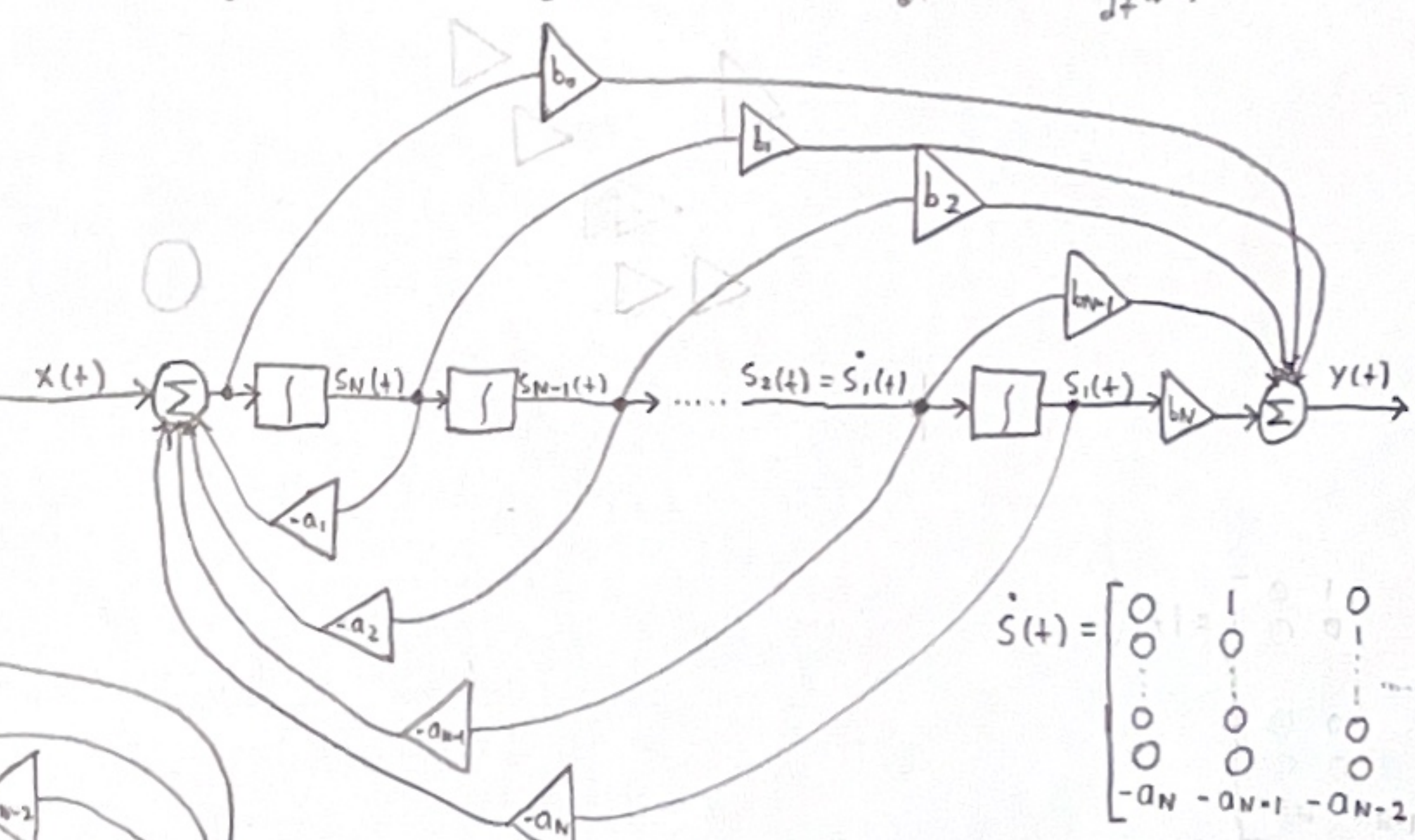
$$y(t) = Ce^{At} \cdot x_0 + C\int_0^t e^{A(t-\tau)} B\,u(\tau)\,d\tau + Du(\tau)$$

$$M_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^{N-1} \end{bmatrix}$$

$\det(M_o) \neq 0 \to$ Observable

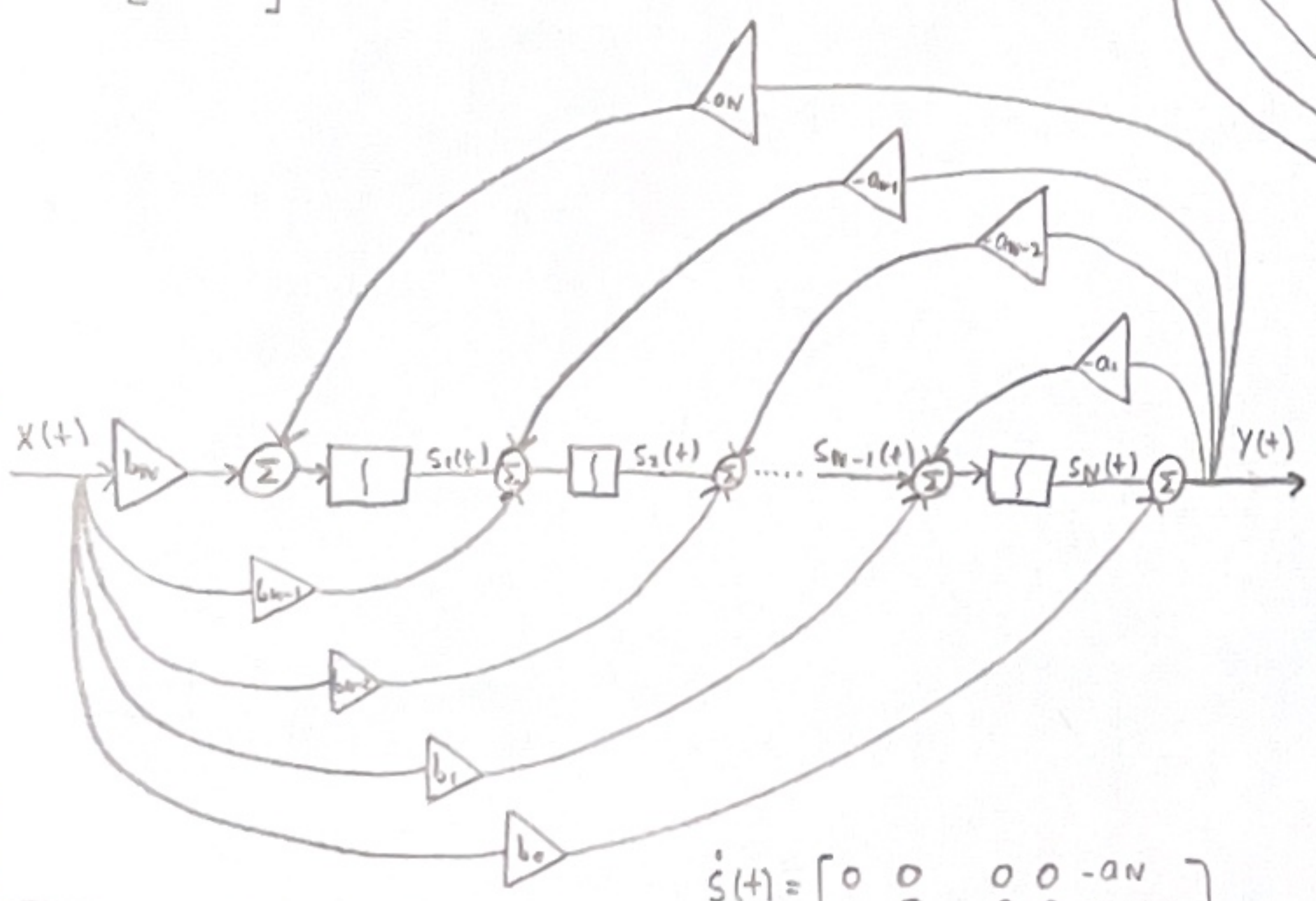
SISO Controllable Canonical Form

ODE: $\dfrac{d^N y(t)}{dt^N} + a_1 \dfrac{d^{N-1} y(t)}{dt^{N-1}} + a_2 \dfrac{d^{N-2} y(t)}{dt^{N-2}} \cdots + a_{N-1}\dfrac{dy(t)}{dt} + a_N y(t) = b_0 \dfrac{d^N x(t)}{dt^N} + b_1 \dfrac{d^{N-1} x(t)}{dt^{N-1}} \cdots + b_{N-1}\dfrac{dx(t)}{dt} + b_N x(t)$



$$\dot{s}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & & 0 & 1 \\ -a_N & -a_{N-1} & -a_{N-2} & \cdots & -a_2 & -a_1 \end{bmatrix} s(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} x(t)$$

$$y(t) = [(b_N - a_N b_0)\ (b_{N-1} - a_{N-1}b_0) \cdots (b_2 - a_2 b_0)\ (b_1 - a_1 b_0)]\,s(t) + b_0 x(t)$$



SISO Observable Canonical Form

ODE: (same as CCF)

$$\dot{s}(t) = \begin{bmatrix} 0 & 0 & & 0 & 0 & -a_N \\ 1 & 0 & & 0 & 0 & -a_{N-1} \\ 0 & 1 & & 0 & 0 & -a_{N-2} \\ \vdots & & \cdots & & & \vdots \\ 0 & 0 & & 1 & 0 & -a_2 \\ 0 & 0 & & 0 & 1 & -a_1 \end{bmatrix} s(t) + \begin{bmatrix} b_N - a_N b_0 \\ b_{N-1} - a_{N-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} x(t)$$

$$y(t) = [0\ 0 \cdots 0\ 1]\,s(t) + b_0 x(t)$$

Diagonalization of a matrix

$A = TDT^{-1}$

$D = T^{-1}AT$

where: $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ *eigenvalues of A

$T = [V_1, V_2]$ *eigenvectors of A

DT CCF/OCF

① $\int \to z^{-1}$

② $x(t) \to x[n]$, $y(t) \to y[n]$

③ $\dot{s}(t) = s[n+1]$

ODE:

$$y[n] + a_1 y[n-1] \ldots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] \ldots + b_N x[n-N]$$

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Table 4.1 Basic Properties of Fourier Series

Basic Properties of Fourier Series		
	Time Domain	Frequency Domain
Signals and constants	$x(t)$, $y(t)$ periodic with period T_0, α, β	X_k, Y_k
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X_k + \beta Y_k$
Parseval's power relation	$P_x = \frac{1}{T_0}\int_{T_0} \lvert x(t)\rvert^2 dt$	$P_x = \sum_k \lvert X_k\rvert^2$
Differentiation	$\frac{dx(t)}{dt}$	$jk\Omega_0 X_k$
Integration	$\int_{-\infty}^{t} x(t')dt'$ only if $X_0 = 0$	$\frac{X_k}{jk\Omega_0}$ $k \neq 0$
Time shifting	$x(t-\alpha)$	$e^{-jk\alpha\Omega_0} X_k$
Frequency shifting	$e^{jM\Omega_0 t}x(t)$	X_{k-M}
Symmetry	$x(t)$ real	$\lvert X_k\rvert = \lvert X_{-k}\rvert$ even function of k; $\angle X_k = -\angle X_{-k}$ odd function of k
Convolution in time	$z(t) = [x*y](t)$	$Z_k = X_k Y_k$

Table 3.2 One-sided Laplace Transforms

	Function of time	Function of s, ROC
(1)	$\delta(t)$	1, whole $s-$plane
(2)	$u(t)$	$\frac{1}{s}$, $\mathcal{Re}[s] > 0$
(3)	$r(t)$	$\frac{1}{s^2}$, $\mathcal{Re}[s] > 0$
(4)	$e^{-at}u(t), a > 0$	$\frac{1}{s+a}$, $\mathcal{Re}[s] > -a$
(5)	$\cos(\Omega_0 t)u(t)$	$\frac{s}{s^2+\Omega_0^2}$, $\mathcal{Re}[s] > 0$
(6)	$\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$, $\mathcal{Re}[s] > 0$
(7)	$e^{-at}\cos(\Omega_0 t)u(t), a > 0$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$, $\mathcal{Re}[s] > -a$
(8)	$e^{-at}\sin(\Omega_0 t)u(t), a > 0$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$, $\mathcal{Re}[s] > -a$
(9)	$2Ae^{-at}\cos(\Omega_0 t+\theta)u(t), a > 0$	$\frac{A\angle\theta}{s+a-j\Omega_0} + \frac{A\angle-\theta}{s+a+j\Omega_0}$, $\mathcal{Re}[s] > -a$
(10)	$\frac{1}{(N-1)!}t^{N-1}u(t)$	$\frac{1}{s^N}$ N an integer, $\mathcal{Re}[s] > 0$
(11)	$\frac{1}{(N-1)!}t^{N-1}e^{-at}u(t)$	$\frac{1}{(s+a)^N}$ N an integer, $\mathcal{Re}[s] > -a$
(12)	$\frac{2A}{(N-1)!}t^{N-1}e^{-at}\cos(\Omega_0 t+\theta)u(t)$	$\frac{A\angle\theta}{(s+a-j\Omega_0)^N} + \frac{A\angle-\theta}{(s+a+j\Omega_0)^N}$, $\mathcal{Re}[s] > -a$

Table 3.1 Basic Properties of One-sided Laplace Transforms

Causal functions and constants	$\alpha f(t), \beta g(t)$	$\alpha F(s), \beta G(s)$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Time shifting	$f(t-\alpha)u(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$tf(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
Second derivative	$\frac{d^2 f(t)}{dt^2}$	$s^2F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt'$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t), \alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert}F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0-) = \lim_{s\to\infty} sF(s)$	

No.	$x_1(t)$	$x_2(t)$	$x_1(t) * x_2(t) = x_2(t) * x_1(t)$	
1	$x(t)$	$\delta(t-T)$	$x(t-T)$	
2	$e^{\lambda t}u(t)$	$u(t)$	$\frac{1-e^{\lambda t}}{-\lambda}u(t)$	
3	$u(t)$	$u(t)$	$tu(t)$	
4	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}u(t)$	$\lambda_1 \neq \lambda_2$
5	$e^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$te^{\lambda t}u(t)$	
6	$te^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$\frac{1}{2}t^2e^{\lambda t}u(t)$	

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Table 5.1 Basic Properties of Fourier Transform

	Time Domain	Frequency Domain
Signals and constants	$x(t), y(t), z(t), \alpha, \beta$	$X(\Omega), Y(\Omega), Z(\Omega)$
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha X(\Omega) + \beta Y(\Omega)$
Expansion/contraction in time	$x(\alpha t)$, $\alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert}X\left(\frac{\Omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\Omega)$
Parseval's energy relation	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\Omega)\rvert^2 d\Omega$
Duality	$X(t)$	$2\pi x(-\Omega)$
Time differentiation	$\frac{d^n x(t)}{dt^n}, n \geq 1$, integer	$(j\Omega)^n X(\Omega)$
Frequency differentiation	$-jtx(t)$	$\frac{dX(\Omega)}{d\Omega}$
Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$
Time shifting	$x(t-\alpha)$	$e^{-j\alpha\Omega}X(\Omega)$
Frequency shifting	$e^{j\Omega_0 t}x(t)$	$X(\Omega - \Omega_0)$
Modulation	$x(t)\cos(\Omega_c t)$	$0.5[X(\Omega - \Omega_c) + X(\Omega + \Omega_c)]$
Periodic signals	$x(t) = \sum_k X_k e^{jk\Omega_0 t}$	$X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$
Symmetry	$x(t)$ real	$\lvert X(\Omega)\rvert = \lvert X(-\Omega)\rvert$; $\angle X(\Omega) = -\angle X(-\Omega)$
Convolution in time	$z(t) = [x * y](t)$	$Z(\Omega) = X(\Omega)Y(\Omega)$
Windowing/Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}[X * Y](\Omega)$
Cosine transform	$x(t)$ even	$X(\Omega) = \int_{-\infty}^{\infty} x(t)\cos(\Omega t)dt$, real
Sine transform	$x(t)$ odd	$X(\Omega) = -j\int_{-\infty}^{\infty} x(t)\sin(\Omega t)dt$, imaginary

Table 5.2 Fourier Transform Pairs

	Function of Time	Function of Ω
(1)	$\delta(t)$	1
(2)	$\delta(t - \tau)$	$e^{-j\Omega\tau}$
(3)	$u(t)$	$\frac{1}{j\Omega} + \pi\delta(\Omega)$
(4)	$u(-t)$	$\frac{-1}{j\Omega} + \pi\delta(\Omega)$
(5)	$\text{sign}(t) = 2[u(t) - 0.5]$	$\frac{2}{j\Omega}$
(6)	$A, -\infty < t < \infty$	$2\pi A\delta(\Omega)$
(7)	$Ae^{-at}u(t), a > 0$	$\frac{A}{j\Omega+a}$
(8)	$Ate^{-at}u(t), a > 0$	$\frac{A}{(j\Omega+a)^2}$
(9)	$e^{-a\lvert t\rvert}, a > 0$	$\frac{2a}{a^2+\Omega^2}$
(10)	$\cos(\Omega_0 t), -\infty < t < \infty$	$\pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$
(11)	$\sin(\Omega_0 t), -\infty < t < \infty$	$-j\pi[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$
(12)	$p(t) = A[u(t+\tau) - u(t-\tau)], \tau > 0$	$2A\tau\frac{\sin(\Omega\tau)}{\Omega\tau}$
(13)	$\frac{\sin(\Omega_0 t)}{\pi t}$	$P(\Omega) = u(\Omega + \Omega_0) - u(\Omega - \Omega_0)$
(14)	$x(t)\cos(\Omega_0 t)$	$0.5[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)]$

Fourier Series of Discrete-time Periodic signals

	$x[n]$ periodic signal of period N	$X[k]$ periodic FS coefficients of period N
Z-transform	$x_1[n] = x[n](u[n] - u[n-N])$	$X[k] = \frac{1}{N}Z(x_1[n])\rvert_{z=e^{j2\pi k/N}}$
DTFT	$x[n] = \sum_k X[k]e^{j2\pi nk/N}$	$X(e^{j\omega}) = \sum_k 2\pi X[k]\delta(\omega - 2\pi k/N)$
LTI response	input $x[n] = \sum_k X[k]e^{j2\pi nk/N}$	output: $y[n] = \sum_k X[k]H(e^{jk\omega_0})e^{j2\pi nk/N}$; $H(e^{j\omega})$ (frequency response of system)
Time-shift (circular shift)	$x[n-M]$	$X[k]e^{-j2\pi kM/N}$
Modulation	$x[n]e^{j2\pi Mn/N}$	$X[k-M]$
Multiplication	$x[n]y[n]$	$\sum_{m=0}^{N-1} X[m]Y[k-m]$ periodic convolution
Periodic convolution	$\sum_{m=0}^{N-1} x[m]y[n-m]$	$NX[k]Y[n]$

Properties of the DTFT

Z-transform:	$x[n], X(z), \lvert z\rvert = 1 \in ROC$	$X(e^{j\omega}) = X(z)\rvert_{z=e^{j\omega}}$
Periodicity:	$x[n]$	$X(e^{j\omega}) = X(e^{j(\omega+2\pi k)})$, k integer
Linearity:	$\alpha x[n] + \beta y[n]$	$\alpha X(e^{j\omega}) + \beta Y(e^{j\omega})$
Time-shifting:	$x[n-N]$	$e^{-j\omega N}X(e^{j\omega})$
Frequency-shift:	$x[n]e^{j\omega_0 n}$	$X(e^{j(\omega-\omega_0)})$
Convolution:	$(x * y)[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication:	$x[n]y[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Symmetry:	$x[n]$ real-valued	$\lvert X(e^{j\omega})\rvert$ even function of ω; $\angle X(e^{j\omega})$ odd function of ω
Parseval's relation:	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2 d\omega$	

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One-sided Z-transforms

Function of Time		Function of z, ROC
(1)	$\delta[n]$	1, Whole z-plane
(2)	$u[n]$	$\frac{1}{1-z^{-1}}$, $\lvert z\rvert > 1$
(3)	$nu[n]$	$\frac{z^{-1}}{(1-z^{-1})^2}$, $\lvert z\rvert > 1$
(4)	$n^2u[n]$	$\frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$, $\lvert z\rvert > 1$
(5)	$\alpha^n u[n]$, $\lvert\alpha\rvert < 1$	$\frac{1}{1-\alpha z^{-1}}$, $\lvert z\rvert > \lvert\alpha\rvert$
(6)	$n\alpha^n u[n]$, $\lvert\alpha\rvert < 1$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$, $\lvert z\rvert > \lvert\alpha\rvert$
(7)	$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > 1$
(8)	$\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$, $\lvert z\rvert > 1$
(9)	$\alpha^n\cos(\omega_0 n)u[n]$, $\lvert\alpha\rvert < 1$	$\frac{1-\alpha\cos(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2z^{-2}}$, $\lvert z\rvert > 1$
(10)	$\alpha^n\sin(\omega_0 n)u[n]$, $\lvert\alpha\rvert < 1$	$\frac{\alpha\sin(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2z^{-2}}$, $\lvert z\rvert > \lvert\alpha\rvert$

Table 10.2 Basic Properties of One-sided Z-transform

Causal signals and constants	$\alpha x[n]$, $\beta y[n]$	$\alpha X(z)$, $\beta Y(z)$
Linearity	$\alpha x[n] + \beta y[n]$	$\alpha X(z) + \beta Y(z)$
Convolution sum	$(x * y)[n] = \sum_k x[n]y[n-k]$	$X(z)Y(z)$
Time shifting – causal	$x[n-N]$ N integer	$z^{-N}X(z)$
Time shifting – non-causal	$x[n-N]$	$z^{-N}X(z) + x[-1]z^{-N+1}$
	$x[n]$ non-causal, N integer	$+x[-2]z^{-N+2} + \cdots + x[-N]$
Time reversal	$x[-n]$	$X(z^{-1})$
Multiplication by n	$nx[n]$	$-z\frac{dX(z)}{dz}$
Multiplication by n^2	$n^2x[n]$	$z^2\frac{d^2X(z)}{dz^2} + z\frac{dX(z)}{dz}$
Finite difference	$x[n] - x[n-1]$	$(1-z^{-1})X(z) - x[-1]$
Accumulation	$\sum_{k=0}^{n} x[k]$	$\frac{X(z)}{1-z^{-1}}$
Initial value	$x[0]$	$\lim_{z\to\infty} X(z)$
Final value	$\lim_{n\to\infty} x[n]$	$\lim_{z\to 1}(z-1)X(z)$

Discrete-time Fourier Transforms (DTFT)

	Discrete-time signal	DTFT $X(e^{j\omega})$, periodic of period 2π
(1)	$\delta[n]$	$1, -\pi \le \omega < \pi$
(2)	A	$2\pi A\delta(\omega), -\pi \le \omega < \pi$
(3)	$e^{j\omega_0 n}$	$2\pi\delta(W-\omega_0), -\pi \le \omega < \pi$
(4)	$\alpha^n u[n], \lvert\alpha\rvert < 1$	$\frac{1}{1-\alpha e^{-j\omega}}, -\pi \le \omega < \pi$
(5)	$n\alpha^n u[n], \lvert\alpha\rvert < 1$	$\frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}, -\pi \le \omega < \pi$
(6)	$\cos(\omega_0 n)u[n]$	$\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)], -\pi \le \omega < \pi$
(7)	$\sin(\omega_0 n)u[n]$	$-j\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)], -\pi \le \omega < \pi$
(8)	$\alpha^{\lvert n\rvert}, \lvert\alpha\rvert < 1$	$\frac{1-\alpha^2}{1-2\alpha\cos(\omega)+\alpha^2}, -\pi \le \omega < \pi$
(9)	$p[n] = u[n+N/2] - u[n-N/2]$	$\frac{\sin(\omega(N+1)/2)}{\sin(\omega/2)}, -\pi \le \omega < \pi$
(10)	$\alpha^n\cos(\omega_0 n)u[n]$	$\frac{1-\alpha\cos(\omega_0)e^{-j\omega}}{1-2\alpha\cos(\omega_0)e^{-j\omega}+\alpha^2e^{-2j\omega}}, -\pi \le \omega < \pi$
(11)	$\alpha^n\sin(\omega_0 n)u[n]$	$\frac{\alpha\sin(\omega_0)e^{-j\omega}}{1-2\alpha\cos(\omega_0)e^{-j\omega}+\alpha^2e^{-2j\omega}}, -\pi \le \omega < \pi$

$\alpha^{n-1}u[n-1] \leftrightarrow \frac{e^{-j\omega}}{1-\alpha e^{-j\omega}}$

No.	$f(t)$	$\mathcal{L}(f)(s) = F(s)$		Reference
1.	1	$\frac{1}{s}$,	$s > 0$	Equation (1.5)
2.	t^n	$\frac{n!}{s^{n+1}}$,	$s > 0$	Equation (1.8)
3.	$\sin at$	$\frac{a}{s^2+a^2}$,	$s > 0$	Example 1.9
4.	$\cos at$	$\frac{s}{s^2+a^2}$,	$s > 0$	Equation (1.10)
5.	e^{at}	$\frac{1}{s-a}$,	$s > a$	Example 1.4
6.	$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$,	$s > a$	Prop. 2.12 with $f = \sin bt$
7.	$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$,	$s > a$	Prop. 2.12 with $f = \cos bt$
8.	t^ne^{at}	$\frac{n!}{(s-a)^{n+1}}$,	$s > a$	Prop. 2.14 with $f = e^{at}$

Discrete Fourier Transform (DFT) (Fourier Series Coefficients)

	$x[n]$ finite-length N aperiodic signal	$\tilde{x}[n]$ periodic extension of period $L \ge N$
IDFT/DFT	$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{L-1}\tilde{X}[k]e^{j2\pi nk/L}$ $x[n] = \tilde{x}[n]W[n], W[n] = u[n] - u[n-N]$	$\tilde{X}[k] = \sum_{n=0}^{L-1}\tilde{x}[n]e^{-j2\pi nk/L}$ $X[k] = \tilde{X}[k]W[n], W[k] = u[k] - u[k-N]$
Circular convolution	$(x \otimes_L y)[n]$	$X[k]Y[k]$
Circular and linear convolution	$(x \otimes_L y)[n] = (x * y)[n], L \ge M+K-1$ $M =$ length of $x[n]$, $K =$ length of $y[n]$	

$\sum_{n=0}^{N} ar^n = \frac{a(1-r^{N+1})}{1-r}$

Euler's formula $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

...for cosine $\cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2}$

...for sine $\sin(\theta) = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

sinc function $\text{sinc}(\theta) := \frac{\sin(\pi\theta)}{\pi\theta}$

6. Sum-Difference Formulas

$\sin(x+y) = \sin x\cos y + \cos x\sin y$ $\quad \tan(x+y) = \frac{\tan x+\tan y}{1-\tan x\tan y}$

$\sin(x-y) = \sin x\cos y - \cos x\sin y$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$ $\quad \tan(x-y) = \frac{\tan x-\tan y}{1+\tan x\tan y}$

$\cos(x-y) = \cos x\cos y + \sin x\sin y$

7. Double Angle Formulas

$\sin(2x) = 2\sin x\cos x$

$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

$\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$

8. Power-Reducing/Half Angle Formulas

$\sin^2 x = \frac{1-\cos 2x}{2}$ $\quad \cos^2 x = \frac{1+\cos 2x}{2}$ $\quad \tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$

$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$ $\quad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$ $\quad \tan\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$

Recap of Transforms

DT

$LT: X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$ $\quad ZT: X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

$ILT: x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{st}ds$ $\quad IZT: x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$

$FT: X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ $\quad DTFT: X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$

$IFT: x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$ $\quad IDTFT: x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$

$FS: X_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$ $\quad DFT: X_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\omega_0 n}$

$IFS: x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_0 t}$ $\quad IDFT: x[n] = \sum_{k=0}^{N-1} X_k e^{jk\omega_0 n}$

where $\omega_0 = \frac{2\pi}{N}$